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COMPLETE SOLUTION TO THE PROBLEM OF AN EXTERNAL CIRCULAR CRACK IN A TRANSVERSELY ISOTROPIC BODY SUBJECTED TO ARBITRARY SHEAR LOADING

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Abstract—A complete solution to the problem of an external circular crack in a transversely isotropic body subjected to arbitrary shear loading is described. Explicit expressions are given for the field of stresses and displacements in a transversely isotropic elastic body weakened by an external circular crack. The crack faces are subjected to arbitrary concentrated shear loading. All the results are presented in terms of elementary functions. The problem of interaction between an arbitrarily located horizontal force and an external circular crack is considered as an example. No similar result seem to have been previously in the literature, even in the case of an isotropic body.

INTRODUCTION

I consider a transversely isotropic elastic space, characterized by the following stress–strain relationships

$$\sigma_x = A_{11}\frac{\partial u_x}{\partial x}+(A_{11}-2A_{66})\frac{\partial u_y}{\partial y}+A_{13}\frac{\partial w}{\partial z},$$

$$\sigma_y = (A_{11}-2A_{66})\frac{\partial u_x}{\partial x}+A_{11}\frac{\partial u_y}{\partial y}+A_{13}\frac{\partial w}{\partial z},$$

$$\sigma_z = A_{13}\frac{\partial u_x}{\partial x}+A_{13}\frac{\partial u_y}{\partial y}+A_{33}\frac{\partial w}{\partial z},$$

$$\tau_{xy} = A_{66}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right),\quad \tau_{yz} = A_{44}\left(\frac{\partial u_y}{\partial z}+\frac{\partial w}{\partial y}\right),$$

$$\tau_{zx} = A_{44}\left(\frac{\partial w}{\partial x}+\frac{\partial u_x}{\partial z}\right). \tag{1}$$

Here A_{ik} are five elastic constants; u_x, u_y and w are the elastic displacements in the Ox, Oy and Oz directions, respectively. The equations of equilibrium are

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}=0,\quad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}=0,\quad \frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}=0. \tag{2}$$

It was shown by Fabrikant (1989) that the general solution of eqns (1) and (2) can be expressed through three potential functions F_1, F_2 and F_3 as follows

$$u = u_x+\mathrm{i}u_y = \Lambda(F_1+F_2+\mathrm{i}F_3),\quad w = m_1\frac{\partial F_1}{\partial z}+m_2\frac{\partial F_2}{\partial z}, \tag{3}$$

where all three functions F_k satisfy the equation (Elliott, 1948)

$$\Delta F_k + \gamma_k^2 \frac{\partial^2 F_k}{\partial z^2} = 0, \quad \text{for } k = 1,2,3. \tag{4}$$

Here

$$\Lambda = \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}, \quad \bar{\Lambda} = \frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Lambda\bar{\Lambda}. \tag{5}$$

The values of m_k and γ_k are related by expressions (Elliott, 1948)

$$\frac{A_{44} + m_k(A_{13} + A_{44})}{A_{11}} = \frac{m_k A_{33}}{m_k A_{44} + A_{13} + A_{44}} = \gamma_k^2, \quad \text{for } k = 1,2$$

$$\gamma_3 = \left(\frac{A_{44}}{A_{66}}\right)^{1/2}. \tag{6}$$

The other elastic constants used in this paper are:

$$G_1 = \beta + \gamma_1\gamma_2 H, \quad G_2 = \beta - \gamma_1\gamma_2 H, \quad \beta = \frac{\gamma_3}{2\pi A_{44}},$$

$$H = \frac{(\gamma_1 + \gamma_2)A_{11}}{2\pi(A_{11}A_{33} - A_{13}^2)}, \quad \alpha = \frac{\sqrt{A_{11}A_{33}} - A_{13}}{A_{11}(\gamma_1 + \gamma_2)}. \tag{7}$$

The field of stresses can be defined through the three potential functions F_k as follows:

$$\sigma_1 = 2A_{66}\frac{\partial^2}{\partial z^2}\{[\gamma_1^2 - (1+m_1)\gamma_3^2]F_1 + [\gamma_2^2 - (1+m_2)\gamma_3^2]F_2\},$$

$$\sigma_2 = 2A_{66}\Lambda^2(F_1 + F_2 + \mathrm{i}F_3),$$

$$\sigma_z = A_{44}\frac{\partial^2}{\partial z^2}[(1+m_1)\gamma_1^2 F_1 + (1+m_2)\gamma_2^2 F_2]$$

$$= -A_{44}\Delta[(1+m_1)F_1 + (1+m_2)F_2],$$

$$\tau_z = A_{44}\Lambda\frac{\partial}{\partial z}[(1+m_1)F_1 + (1+m_2)F_2 + \mathrm{i}F_3]. \tag{8}$$

Here the following useful combinations of stress components are introduced:

$$\sigma_1 = \sigma_x + \sigma_y, \quad \sigma_2 = \sigma_x - \sigma_y + 2\mathrm{i}\tau_{xy}, \quad \tau_z = \tau_{zx} + \mathrm{i}\tau_{yz}. \tag{9}$$

The general solution (3) and (8) was used in obtaining the complete solution to the penny-shaped crack problem (Fabrikant, 1989). To the best of my knowledge, the complete solution to the problem of an external crack under shear loading has not been reported in the literature, even in the case of an isotropic body. It is considered in the next section.

FORMULATION OF THE PROBLEM AND SOLUTION OF THE GOVERNING INTEGRAL EQUATION

Consider a transversely isotropic elastic space weakened by an external circular crack of radius a in the plane $z = 0$. Let arbitrary shear τ_z be applied to crack faces. Owing to symmetry, the problem can be reduced to the mixed boundary value problem for an elastic half-space $z \geqslant 0$, subject to the following conditions on the plane $z = 0$:

$$
\begin{aligned}
\tau_z &= -\tau(\rho,\phi), &&\text{for } a \leqslant \rho < \infty, && 0 \leqslant \phi < 2\pi,\\
u &= 0, &&\text{for } 0 \leqslant \rho \leqslant a, && 0 \leqslant \phi < 2\pi,\\
\sigma_z &= 0, &&\text{for } 0 \leqslant \rho < \infty, && 0 \leqslant \phi < 2\pi.
\end{aligned}
\tag{10}
$$

The general solution through three potential functions F_k was given by Fabrikant (1989: Section 4.4) as

$$
\begin{aligned}
F_1 &= -\frac{1}{4\pi(m_1-1)}(\Lambda\bar{\chi}_1 + \bar{\Lambda}\chi_1),\\
F_2 &= -\frac{1}{4\pi(m_2-1)}(\Lambda\bar{\chi}_2 + \bar{\Lambda}\chi_2),\\
F_3 &= \frac{1}{4\pi}(\Lambda\bar{\chi}_3 - \bar{\Lambda}\chi_3),
\end{aligned}
\tag{11}
$$

where $\chi_k(x,y,z)$ is understood as $\chi(x,y,z_k)$, and $z_k = z/\gamma_k$. As we see from eqn (11), the complete solution is expressed through just *one* complex harmonic function $\chi(x,y,z)$. This function is related to crack surface displacements u by (Fabrikant, 1989: formula 4.4.9)

$$
\chi(\rho,\phi,z) = \int_0^{2\pi}\int_a^{\infty} \ln[\sqrt{\rho^2+r^2-2\rho r\cos(\phi-\psi)+z^2}+z]u(r,\psi)r\,\mathrm{d}r\,\mathrm{d}\psi. \tag{12}
$$

The governing integro-differential equation, which relates the unknown crack face displacements u to the prescribed shear loading τ, is derived by Fabrikant (1989), and is:

$$
\frac{1}{2\pi^2(G_1^2-G_2^2)}\left[G_1\Delta\iint_S \frac{u(N)}{R(N,N_0)}\,\mathrm{d}S_N + G_2\Lambda^2\iint_S \frac{\bar{u}(N)}{R(N,N_0)}\,\mathrm{d}S_N\right] = -\tau(N_0). \tag{13}
$$

Here points N and N_0 have the polar cylindrical coordinates $(r,\psi,0)$ and $(\rho_0,\phi_0,0)$, respectively, $R(N,N_0)$ stands for the distance between the two points, G_1 and G_2 are the elastic constants defined in eqn (7), Δ and Λ are the operators defined in eqn (5), S is the domain of the crack, in this particular case it is the exterior of the circle $\rho = a$.

Equation (13) was solved for the case of a penny-shaped crack in Fabrikant (1989: Section 2.7). To the best of my knowledge, no solution of eqn (13) for the case of an external circular crack has been published so far. Such a solution can be obtained from the continuity expression which defines the shear tractions in the crack neck directly in terms of the prescribed loading τ. Such an expression can be derived from the results in Fabrikant (1989: Section 2.6), namely,

$$
\begin{aligned}
\tau^{(1)}(\rho,\phi) = \frac{1}{\pi^2\sqrt{a^2-\rho^2}}\Bigg\{&\int_0^{2\pi}\int_a^{\infty}\frac{\sqrt{\rho_0^2-a^2}}{R^2}\tau(\rho_0,\phi_0)\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0\\
&+\frac{G_2}{G_1}\int_0^{2\pi}\int_a^{\infty}\frac{\sqrt{\rho_0^2-a^2}(1+\bar{\zeta})\,\mathrm{e}^{2i\phi_0}}{\rho_0^2(1-\bar{\zeta})^2}\bar{\tau}(\rho_0,\phi_0)\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0\Bigg\}.
\end{aligned}
\tag{14}
$$

Here

$$
R^2 = \rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0), \quad \zeta = \frac{\rho}{\rho_0}\mathrm{e}^{i(\phi-\phi_0)}. \tag{15}
$$

The main value of eqn (14) lies in the fact that now the shear stress is known all over the plane $z = 0$, so the solution of eqn (13) can be formally written as (Fabrikant, 1989: Section 2.6)

$$u(\rho,\phi) = \tfrac{1}{2}G_1\int_0^{2\pi}\int_0^a \frac{\tau^{(i)}(\rho_0,\phi_0)}{R}\rho_0\,d\rho_0\,d\phi_0 + \tfrac{1}{2}G_2\int_0^{2\pi}\int_0^a \frac{q\bar{\tau}^{(i)}(\rho_0,\phi_0)}{\bar{q}R}\rho_0\,d\rho_0\,d\phi_0$$
$$+\tfrac{1}{2}G_1\int_0^{2\pi}\int_a^\infty \frac{\tau(\rho_0,\phi_0)}{R}\rho_0\,d\rho_0\,d\phi_0 + \tfrac{1}{2}G_2\int_0^{2\pi}\int_a^\infty \frac{q\bar{\tau}(\rho_0,\phi_0)}{\bar{q}R}\rho_0\,d\rho_0\,d\phi_0. \quad (16)$$

Here the following notation is introduced:

$$q = \rho\,e^{i\phi} - \rho_0\,e^{i\phi_0}, \quad R^2 = q\bar{q}. \quad (17)$$

Substitution of eqn (14) in the first two terms of eqn (16), interchange of the order of integration, with subsequent computation of the relevant integrals, leads to the following solution of eqn (13):

$$u(\rho,\phi) = \frac{G_1}{\pi}\int_0^{2\pi}\int_a^\infty\left[\frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right) - \frac{G_2^2}{G_1^2}\frac{t^2(1+t)}{a^2(1-t)^2}\eta\right]\tau(\rho_0,\phi_0)\rho_0\,d\rho_0\,d\phi_0$$
$$+\frac{G_2}{\pi}\int_0^{2\pi}\int_a^\infty\left[\frac{q}{\bar{q}R}\tan^{-1}\left(\frac{\eta}{R}\right) + \frac{\eta}{\bar{q}}\left(\frac{t\,e^{i\phi}}{\rho(1-t)} - \frac{\bar{t}\,e^{i\phi_0}}{\rho_0(1-\bar{t})}\right)\right]\bar{\tau}(\rho_0,\phi_0)\rho_0\,d\rho_0\,d\phi_0, \quad (18)$$

where

$$\eta = \frac{\sqrt{\rho^2-a^2}\sqrt{\rho_0^2-a^2}}{a}, \quad t = \frac{a^2}{\rho\rho_0}e^{i(\phi-\phi_0)}. \quad (19)$$

The derivation of eqn (18) requires computation of four integrals, namely,

$$\int_0^{2\pi}\int_0^a \frac{\sqrt{\rho_0^2-a^2}}{\sqrt{a^2-r^2}}\frac{1}{\sqrt{\rho^2+r^2-2\rho r\cos(\phi-\psi)}}\frac{r\,dr\,d\psi}{\rho_0^2+r^2-2\rho_0 r\cos(\phi_0-\psi)}$$
$$= \frac{\pi^2}{R}\left[1-\frac{2}{\pi}\tan^{-1}\frac{\sqrt{\rho^2-a^2}\sqrt{\rho_0^2-a^2}}{aR}\right], \quad (20)$$

$$\int_0^{2\pi}\int_0^a \frac{\sqrt{\rho_0^2-a^2}}{\sqrt{a^2-r^2}}\frac{\rho\,e^{i\phi}-r\,e^{i\psi}}{\rho\,e^{-i\phi}-r\,e^{-i\psi}}\frac{1}{\sqrt{\rho^2+r^2-2\rho r\cos(\phi-\psi)}}\frac{r\,dr\,d\psi}{\rho_0^2+r^2-2\rho_0 r\cos(\phi_0-\psi)}$$
$$= 2\pi\left\{\frac{q}{\bar{q}R}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{\eta}{R}\right)\right] + \frac{\eta}{\bar{q}}\left[\frac{\bar{t}\,e^{i\phi_0}}{\rho_0(1-\bar{t})} - \frac{t\,e^{i\phi}}{\rho(1-t)}\right]\right\}$$
$$-2\pi\sqrt{\rho_0^2-a^2}\,\frac{e^{2i\phi_0}}{\rho_0^2}\int_0^a \frac{(1+\xi)\,dx}{\sqrt{\rho^2-x^2(1-\xi)^2}}, \quad (21)$$

$$\int_0^{2\pi}\int_0^a \frac{\sqrt{\rho_0^2-a^2}}{\sqrt{a^2-r^2}}\frac{e^{2i\phi_0}(\rho_0+r\,e^{-i(\psi-\phi_0)})}{\rho_0(\rho_0-r\,e^{-i(\psi-\phi_0)})^2}\frac{r\,dr\,d\psi}{\sqrt{\rho^2+r^2-2\rho r\cos(\phi-\psi)}}$$
$$= 2\pi\sqrt{\rho_0^2-a^2}\,\frac{e^{2i\phi_0}}{\rho_0^2}\int_0^a \frac{(1+\bar{\xi})\,dx}{\sqrt{\rho^2-x^2(1-\bar{\xi})^2}}, \quad (22)$$

$$\int_0^{2\pi}\int_0^a \frac{\sqrt{\rho_0^2-a^2}}{\sqrt{a^2-r^2}}\,\frac{e^{-2i\phi_0}(\rho_0+r\,e^{i(\psi-\phi_0)})}{\rho_0(\rho_0-r\,e^{i(\psi-\phi_0)})^2}\,\frac{\rho\,e^{i\phi}-r\,e^{i\psi}}{\rho\,e^{-i\phi}-r\,e^{-i\psi}}\,\frac{r\,dr\,d\psi}{\sqrt{\rho^2+r^2-2\rho r\cos(\phi-\psi)}}$$
$$= 2\pi\eta\frac{t^2(1+t)}{a^2(1-t)^2}, \qquad (23)$$

where q and R are given in eqn (17), t and η are defined by eqn (19), and

$$\xi = \frac{x^2}{\rho\rho_0}\,e^{i(\phi-\phi_0)}. \qquad (24)$$

The methods developed by Fabrikant (1989) were used in computation of eqns (20)–(23). I note that the integral in eqn (22) is easily computable in terms of elementary functions. I did not perform this computation because eqn (22), after substitution in (18) cancels out with the last term in (21).

Now expression (18) gives the exact closed form solution to the governing integro-differential equation (13). The substitution of (18) in (12) allows one to compute the main potential function χ which, in turn, defines all three functions F_k in eqn (11); and, finally, the substitution of F_k in eqns (3) and (8) will give us the complete solution for the field of displacements and stresses, respectively. Owing to the complexity of the integrals involved, the procedure is very non-trivial, and it will be described in detail in the next section.

THE COMPLETE SOLUTION

The direct substitution of eqn (18) in (12), interchange of the order of integration, and computation of the relevant integrals does not seem possible at first sight. Certain properties of the integrands in eqn (18) need to be explained, which will prove useful in the future computation of the integrals involved.

I note the following property

$$\Lambda\left[\frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right)-\frac{\eta\bar{t}^2(1+\bar{t})}{a^2(1-\bar{t})^2}\right] = -\bar{\Lambda}\left[\frac{q}{\bar{q}R}\tan^{-1}\left(\frac{\eta}{R}\right)-\frac{\eta}{\bar{q}}\left(\frac{t\,e^{i\phi}}{\rho(1-t)}-\frac{\bar{t}\,e^{i\phi_0}}{\rho_0(1-\bar{t})}\right)\right]. \qquad (25)$$

I introduce the following notation:

$$B_1(N,N_0) = \frac{1}{R(N,N_0)}\tan^{-1}\left(\frac{\sqrt{r^2-a^2}\sqrt{\rho_0^2-a^2}}{aR(N,N_0)}\right),$$

$$B_2(N,N_0) = \frac{a\sqrt{r^2-a^2}\sqrt{\rho_0^2-a^2}(r\rho_0\,e^{i(\psi-\phi_0)}+a^2)}{r\rho_0\,e^{i(\psi-\phi_0)}(r\rho_0\,e^{i(\psi-\phi_0)}-a^2)^2},$$

$$B_3(N,N_0) = \frac{r\,e^{i\psi}-\rho_0\,e^{i\phi_0}}{(r\,e^{-i\psi}-\rho_0\,e^{-i\phi_0})R(N,N_0)}\tan^{-1}\left(\frac{\sqrt{r^2-a^2}\sqrt{\rho_0^2-a^2}}{aR(N,N_0)}\right)$$
$$+\frac{a\sqrt{r^2-a^2}\sqrt{\rho_0^2-a^2}}{r\,e^{-i\psi}-\rho_0\,e^{-i\phi_0}}\left[\frac{e^{i\psi}}{r(r\rho_0\,e^{i(\psi-\phi_0)}-a^2)}-\frac{e^{i\phi_0}}{\rho_0(r\rho_0\,e^{i(\psi-\phi_0)}-a^2)}\right]. \qquad (26)$$

Here the points N and N_0 are characterized by the polar cylindrical coordinates $(r, \psi, 0)$ and $(\rho_0, \phi_0, 0)$, respectively.

I note the following property of symmetry

$$B_1(N,N_0)=B_1(N_0,N),\quad B_2(N,N_0)=\bar{B}_2(N_0,N),$$
$$B_3(N,N_0)=B_3(N_0,N). \tag{27}$$

Let $R(M,N)$ denote the distance between the points $M(\rho,\phi,z)$ and $N(r,\psi,0)$.
By using eqns (25) and (26),

$$\iint_S \Lambda[B_1(N,N_0)-B_2(N,N_0)]\frac{dS_N}{R(M,N)}=-\iint_S \bar{\Lambda}B_3(N,N_0)\frac{dS_N}{R(M,N)}. \tag{28}$$

Here S is the domain of the crack. Integration by parts in eqn (28) leads to a very important property

$$\iint_S [B_1(N,N_0)-B_2(N,N_0)]\Lambda\left(\frac{1}{R(M,N)}\right)dS_N=-\iint_S B_3(N,N_0)\bar{\Lambda}\left(\frac{1}{R(M,N)}\right)dS_N. \tag{29}$$

Two more properties can be obtained by application of Λ and $\bar{\Lambda}$ to both sides of eqn (29), namely

$$\iint_S [B_1(N,N_0)-B_2(N,N_0)]\Lambda^2\left(\frac{1}{R(M,N)}\right)dS_N=-\iint_S B_3(N,N_0)\Delta\left(\frac{1}{R(M,N)}\right)dS_N,$$
$$\iint_S [B_1(N,N_0)-B_2(N,N_0)]\Delta\left(\frac{1}{R(M,N)}\right)dS_N=-\iint_S B_3(N,N_0)\bar{\Lambda}^2\left(\frac{1}{R(M,N)}\right)dS_N. \tag{30}$$

Integration of both sides in eqns (29) and (30) with respect to z will lead to similar properties for $\ln[R(M,N)+z]$ integrand. These properties allow one to avoid computation of integrals involving B_3, which look very formidable, and compute instead the integrals involving expressions B_1 and B_2 which are more simple.

It can be inferred from eqn (11) that it will be useful to introduce the notation:

$$U=\Lambda\bar{\chi}+\bar{\Lambda}\chi,\quad V=\Lambda\bar{\chi}-\bar{\Lambda}\chi. \tag{31}$$

The complete solution, given by eqns (11), (3) and (8) will depend only on the first and second derivatives of U and V. Since integrals involving B_3 do not have to be evaluated owing to the properties (28)–(30), all the derivatives of U and V can be expressed through the two fundamental functions, namely,

$$L_1(M,N_0)=\iint_S B_1(N,N_0)\ln[R(M,N)+z]\,dS_N,$$
$$L_2(M,N_0)=\iint_S B_2(N,N_0)\ln[R(M,N)+z]\,dS_N. \tag{32}$$

Equation (18) can be rewritten in the new notation as

$$u(N)=\frac{G_1}{\pi}\iint_S\left[B_1(N,N_0)-\frac{G_2^2}{G_1^2}\bar{B}_2(N,N_0)\right]\tau(N_0)\,dS_{N_0}$$
$$+\frac{G_2}{\pi}\iint_S B_3(N,N_0)\bar{\tau}(N_0)\,dS_{N_0}. \tag{33}$$

By substituting eqns (33) and (12) in (31) and using the properties (28)–(30), the following results are obtained:

$$U(M)=\frac{G_1-G_2}{\pi}\left\{\bar{\Lambda}\iint_S\left[L_1(M,N_0)+\frac{G_2}{G_1}\bar{L}_2(M,N_0)\right]\tau(N_0)\,\mathrm{d}S_{N_0}\right.$$
$$\left.-\Lambda\iint_S\left[L_1(M,N_0)+\frac{G_2}{G_1}L_2(M,N_0)\right]\bar{\tau}(N_0)\,\mathrm{d}S_{N_0}\right\},$$
$$V(M)=\frac{G_1+G_2}{\pi}\left\{-\bar{\Lambda}\iint_S\left[L_1(M,N_0)-\frac{G_2}{G_1}\bar{L}_2(M,N_0)\right]\tau(N_0)\,\mathrm{d}S_{N_0}\right.$$
$$\left.+\Lambda\iint_S\left[L_1(M,N_0)-\frac{G_2}{G_1}L_2(M,N_0)\right]\bar{\tau}(N_0)\,\mathrm{d}S_{N_0}\right\}. \quad (34)$$

In order to find the field of displacements, one need only know the Λ and z-derivatives of U and V; the field of stresses will be completely defined by the second Λ, z and mixed derivatives. All these derivatives can be expressed in elementary functions, as will be shown in the next section.

THE GREEN'S FUNCTIONS

The results of previous section can be applied to solving the problem of a tangential point force loading of an external circular crack. The solution will give all the Green's functions related to the case. I consider an infinite transversely isotropic solid weakened in the plane $z=0$ by an external circular crack $\rho \geqslant a$.

Let two equal and oppositely directed tangential forces of magnitude $T=T_x+\mathrm{i}T_y$ be applied to the crack faces at the points $(\rho_0, \phi_0, 0^{\pm})$. I show in some detail computation of the tangential displacement u, which is defined by the first formula in (3), with the functions F_k given in (11). From eqns (3), (11), (31) and (34) it can be deduced that only ΔL_1, ΔL_2, $\Lambda^2 L_1$ and $\Lambda^2 L_2$ need to be computed. Since both L_1 and L_2 are harmonic functions of (ρ, ϕ, z), computation of Δ can be replaced by computation of $-\partial^2/\partial z^2$. The functions F_k defined by eqn (11) can be rewritten in terms of U and V as follows

$$F_1=-\frac{U_1}{4\pi(m_1-1)},\quad F_2=-\frac{U_2}{4\pi(m_2-1)},\quad F_3=\frac{\mathrm{i}V_3}{4\pi}. \quad (35)$$

Here U_k and V_k are understood as $U(M_k)$ and $V(M_k)$, and the point M_k has the coordinates (ρ, ϕ, z_k), with $z_k=z/\gamma_k$, for $k=1,2,3$. From eqns (3) and (34)–(35) it may be concluded that

$$u=\frac{G_1-G_2}{4\pi^2}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{\frac{\partial^2}{\partial z_k^2}\left[L_1(M_k,N_0)+\frac{G_2}{G_1}\bar{L}_2(M_k,N_0)\right]T\right.$$
$$\left.-\Lambda^2\left[L_1(M_k,N_0)+\frac{G_2}{G_1}L_2(M_k,N_0)\right]\bar{T}\right\}$$
$$-\frac{G_1+G_2}{4\pi^2}\left\{\frac{\partial^2}{\partial z_3^2}\left[L_1(M_3,N_0)-\frac{G_2}{G_1}\bar{L}_2(M_3,N_0)\right]T\right.$$
$$\left.+\Lambda^2\left[L_1(M_3,N_0)-\frac{G_2}{G_1}L_2(M_3,N_0)\right]\bar{T}\right\}. \quad (36)$$

The second z-derivatives of L_1 and L_2 are computed in formulae (A37) and (A11) of Appendix A, the quantities of $\Lambda^2 L_1$ and $\Lambda^2 L_2$ are given in (A34) and (A42), respectively.

So, utilization of (A37), (A11), (A34) and (A42) in (36) gives the complete field of tangential displacements in the whole space weakened by an external crack and subjected to a pair of tangential forces T applied at the points N_0 of the crack faces. All the remaining quantities can be computed in a similar manner, with all the necessary derivatives of L_1 and L_2 presented in Appendix A. The final results are

$$u = \frac{G_1 - G_2}{2\pi} \sum_{k=1}^{2} \frac{1}{m_k - 1} \left\{ -\left[g_2(z_k) + \frac{G_2}{G_1} \bar{g}_7(z_k) \right] T + \left[g_{16}(z_k) + \frac{G_2}{G_1} g_8(z_k) \right] \bar{T} \right\}$$
$$+ \frac{G_1 + G_2}{2\pi} \left\{ \left[g_2(z_3) - \frac{G_2}{G_1} \bar{g}_7(z_3) \right] T + \left[g_{16}(z_3) - \frac{G_2}{G_1} g_8(z_3) \right] \bar{T} \right\}, \quad (37)$$

$$w = \frac{2}{\pi} H \gamma_1 \gamma_2 \, \mathrm{Re} \left\{ \sum_{k=1}^{2} \frac{m_k}{(m_k - 1)\gamma_k} \left[\bar{g}_1(z_k) + \frac{G_2}{G_1} \bar{g}_9(z_k) \right] T \right\}, \quad (38)$$

$$\sigma_1 = \mathrm{Re} \left\{ \frac{2\gamma_1 \gamma_2}{\pi^2(\gamma_1 - \gamma_2)} \sum_{k=1}^{2} (-1)^{k+1} \left[\frac{1}{\gamma_3^2 (m_k + 1)} - \frac{1}{\gamma_k^2} \right] \left[\bar{g}_5(z_k) + \frac{G_2}{G_1} \bar{g}_{10}(z_k) \right] T \right\}, \quad (39)$$

$$\sigma_2 = -\frac{2}{\pi} A_{66} H \gamma_1 \gamma_2 \sum_{k=1}^{2} \frac{1}{m_k - 1} \left\{ \left[g_5(z_k) + \frac{G_2}{G_1} \bar{g}_{13}(z_k) \right] T \right.$$
$$\left. + \left[g_{11}(z_k) + \frac{G_2}{G_1} g_{12}(z_k) \right] \bar{T} \right\} - \frac{1}{\pi^2 \gamma_3} \left\{ \left[-g_5(z_3) + \frac{G_2}{G_1} \bar{g}_{13}(z_3) \right] T \right.$$
$$\left. + \left[g_{11}(z_3) - \frac{G_2}{G_1} g_{12}(z_3) \right] \bar{T} \right\}, \quad (40)$$

$$\sigma_z = \mathrm{Re} \left\{ \frac{\gamma_1 \gamma_2}{\pi^2(\gamma_1 - \gamma_2)} \sum_{k=1}^{2} (-1)^{k+1} \left[\bar{g}_5(z_k) + \frac{G_2}{G_1} \bar{g}_{10}(z_k) \right] T \right\}, \quad (41)$$

$$\tau_z = \frac{\gamma_1 \gamma_2}{2\pi^2(\gamma_1 - \gamma_2)} \sum_{k=1}^{2} \frac{(-1)^k}{\gamma_k} \left\{ \left[g_3(z_k) + \frac{G_2}{G_1} \bar{g}_{14}(z_k) \right] T \right.$$
$$\left. + \left[-g_4(z_k) + \frac{G_2}{G_1} g_{15}(z_k) \right] \bar{T} \right\} + \frac{1}{2\pi^2} \left\{ \left[g_3(z_3) - \frac{G_2}{G_1} \bar{g}_{14}(z_3) \right] T \right.$$
$$\left. + \left[g_4(z_3) + \frac{G_2}{G_1} g_{15}(z_3) \right] \bar{T} \right\}. \quad (42)$$

Here Re stands for the real part of the expression to follow, the elastic coefficients are defined in (6) and (7) and the functions g_k are given by (for details see Appendix A)

$$g_1(z) = \frac{1}{\bar{q}} \left\{ \tan^{-1} \frac{(\rho_0^2 - a^2)^{1/2}}{a} - \frac{z}{R_0} \tan^{-1} \left(\frac{j}{R_0} \right) \right.$$
$$\left. + \frac{(\rho_0^2 - a^2)^{1/2}}{\bar{s}} \left[\tan^{-1} \left(\frac{\bar{s}}{(a^2 - l_1^2)^{1/2}} \right) - \tan^{-1} \left(\frac{\bar{s}}{a} \right) \right] \right\}, \quad (43)$$

$$g_2(z) = \frac{1}{R_0} \tan^{-1} \left(\frac{j}{R_0} \right), \quad (44)$$

$$g_3(z) = -\frac{z}{R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right)+\frac{j}{z(R_0^2+j^2)}\left[\frac{l_2^2-\rho^2}{l_2^2-l_1^2}-\frac{z^2}{R_0^2}\right], \tag{45}$$

$$\begin{aligned} g_4(z) = \frac{1}{\bar{q}}\Bigg\{&\frac{z(3R_0^2-z^2)}{\bar{q}R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right)-\frac{2}{\bar{q}}\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right)+\frac{zj}{R_0^2+j^2}\left[\frac{q}{R_0^2}\right.\\ &\left.-\frac{\bar{q}\rho^2 e^{2i\phi}}{(l_2^2-l_1^2)(\rho^2-l_1^2)}\right]-\frac{(\rho_0^2-a^2)^{1/2}}{\bar{s}}\left(\frac{2}{\bar{q}}+\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}\right)\left[\tan^{-1}\left(\frac{\bar{s}}{(a^2-l_1^2)^{1/2}}\right)\right.\\ &\left.-\tan^{-1}\left(\frac{\bar{s}}{a}\right)\right]+\frac{(\rho_0^2-a^2)^{1/2}}{\bar{s}^2}\left[\frac{(a^2-l_1^2)^{1/2}\rho_0 e^{i\phi_0}}{\rho\rho_0 e^{-i(\phi-\phi_0)}-l_1^2}-\frac{a e^{i\phi}}{\rho}\right]\Bigg\}, \end{aligned} \tag{46}$$

$$g_5(z) = -\frac{q}{R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right)+\frac{j}{R_0^2+j^2}\left[\frac{\rho e^{i\phi}}{l_2^2-l_1^2}-\frac{q}{R_0^2}\right], \tag{47}$$

$$\begin{aligned} g_7(z) = \frac{z}{a^3}(\rho_0^2-a^2)^{1/2}\Bigg\{&\frac{3\bar{t}^{5/2}}{(1-\bar{t})^{5/2}}\left[\tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-l_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right)\right.\\ &\left.-\tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right)\right]+\frac{\bar{t}^2}{(1-\bar{t})^2}\left[\frac{a(a^2-l_1^2)^{1/2}}{a^2-l_1^2\bar{t}}-2-\bar{t}+\frac{a(1+\bar{t})}{(a^2-l_1^2)^{1/2}}\right]\Bigg\}, \end{aligned} \tag{48}$$

$$\begin{aligned} g_8(z) = \frac{\bar{t}^2}{a^2}\rho^2 e^{2i\phi}(\rho_0^2-a^2)^{1/2}\Bigg[&\frac{a}{(a^2-\rho^2\bar{t})^{3/2}}\tan^{-1}\left(\frac{(a^2-\rho^2\bar{t})^{1/2}}{(l_2^2-a^2)^{1/2}}\right)\\ &-\frac{\bar{t}l_1(\rho^2-l_1^2)^{1/2}}{(a^2-l_1^2\bar{t})(a^2-\rho^2\bar{t})}\Bigg], \end{aligned} \tag{49}$$

$$\begin{aligned} g_9(z) = -\frac{\bar{t}^2}{a^3}\rho e^{i\phi}(\rho_0^2-a^2)^{1/2}\Bigg\{&\frac{1}{1-\bar{t}}\left[1-\frac{a(a^2-l_1^2)^{1/2}}{a^2-l_1^2\bar{t}}\right]\\ &+\frac{\bar{t}^{1/2}}{(1-\bar{t})^{3/2}}\left[\tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-l_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right)\right]\Bigg\}, \end{aligned} \tag{50}$$

$$g_{10}(z) = \frac{j\rho e^{i\phi}\bar{t}^2 l_1^2(a^2+l_1^2\bar{t})}{a^2(a^2-l_1^2\bar{t})^2(l_2^2-l_1^2)}, \tag{51}$$

$$\begin{aligned} g_{11}(z) = \frac{1}{\bar{q}}\Bigg\{&\frac{3R_0^4+6z^2R_0^2-z^4}{\bar{q}^2R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right)-\frac{8z}{\bar{q}^2}\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right)\\ &-(\rho_0^2-a^2)^{1/2}\left[-\frac{3(1-\zeta)^{1/2}}{\bar{q}^2}\tan^{-1}\left(\frac{a(1-\zeta)^{1/2}}{(l_2^2-a^2)^{1/2}}\right)+\frac{z}{\bar{s}}\left(\frac{8}{\bar{q}^2}+\frac{4\rho_0 e^{i\phi_0}}{\bar{q}\bar{s}^2}\right.\right.\\ &\left.+\frac{3\rho_0^2 e^{2i\phi_0}}{\bar{s}^4}\right)\left[\tan^{-1}\left(\frac{\bar{s}}{(a^2-l_1^2)^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{s}}{a}\right)\right]-\frac{z}{\bar{s}^2}\left(\frac{2}{\bar{q}}\right.\\ &\left.\left.+\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}\right)\left(\frac{(a^2-l_1^2)^{1/2}\rho_0 e^{i\phi_0}}{\rho\rho_0 e^{-i(\phi-\phi_0)}-l_1^2}-\frac{a e^{i\phi}}{\rho}\right)\right]+\frac{2ja e^{i\phi}}{\rho\bar{s}^2}\left(\frac{1}{\bar{q}}+\frac{e^{i\phi}}{\rho}+\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}\right)\\ &\times(a-(a^2-l_1^2)^{1/2})+\frac{j}{R_0^2+j^2}\left[\frac{\rho e^{3i\phi}\bar{q}}{l_2^2-l_1^2}+\frac{z^2q}{R_0^2\bar{q}}+\frac{e^{i\phi}(\rho^2-l_1^2)}{\bar{q}\rho}-2e^{2i\phi}\right]\Bigg\}, \end{aligned} \tag{52}$$

$$g_{12}(z) = \frac{\bar{t}^3}{a}\rho^3 e^{3i\phi}(\rho_0^2-a^2)^{1/2}\left\{\frac{(l_2^2-a^2)^{1/2}}{(a^2-l_1^2\bar{t})(a^2-\rho^2\bar{t})}\left[\frac{a^2+2\rho^2\bar{t}}{l_2^2(a^2-\rho^2\bar{t})}\right.\right.$$

$$\left.\left.+\frac{a^2+\rho^2\bar{t}}{l_2^2\bar{t}(l_2^2-l_1^2)}+\frac{2(a^2-l_1^2)}{(l_2^2-l_1^2)(a^2-l_1^2\bar{t})}\right]-\frac{3}{(a^2-\rho^2\bar{t})^{5/2}}\tan^{-1}\left(\frac{(a^2-\rho^2\bar{t})^{1/2}}{(l_2^2-a^2)^{1/2}}\right)\right\}, \quad (53)$$

$$\bar{g}_{13}(z) = -zt^2\rho e^{i\phi}\frac{(\rho_0^2-a^2)^{1/2}}{a^3}\left\{\frac{\rho^4(l_2^2+\rho^2 t)(l_2^2-a^2)}{l_2(l_2^2-\rho^2 t)^2(l_2^2-\rho^2)^{3/2}(l_2^2-l_1^2)}\right.$$

$$-\frac{15t^{1/2}}{\rho^2(1-t)^{7/2}}\left[\tan^{-1}\left(\frac{t^{1/2}}{(1-t)^{1/2}}\right)-\tan^{-1}\left(\frac{t^{1/2}(a^2-l_1^2)^{1/2}}{a(1-t)^{1/2}}\right)\right]$$

$$-\frac{1}{(1-t)^2}\left[\frac{2(1+t)}{\rho^2}+\frac{6+9t}{\rho^2(1-t)}\right]+\frac{a}{(a^2-l_1^2)^{1/2}(1-t)^2}\left[\frac{2(1+t)}{\rho^2}\right.$$

$$\left.\left.+\frac{6+9t}{\rho^2(1-t)}-\frac{1+t}{l_2^2-\rho^2}-\frac{3}{l_2^2-\rho^2 t}\right]\right\}, \quad (54)$$

$$g_{14}(z) = -\bar{t}^2\frac{(\rho_0^2-a^2)^{1/2}}{a^3}\left\{\frac{3\bar{t}^{1/2}}{(1-\bar{t})^{5/2}}\left[\tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right)\right.\right.$$

$$\left.-\tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-l_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right)\right]-\frac{1}{(1-\bar{t})^2}\left[\frac{a(a^2-l_1^2)^{1/2}}{a^2-l_1^2\bar{t}}-2-\bar{t}+\frac{a(1+\bar{t})}{(a^2-l_1^2)^{1/2}}\right]$$

$$\left.+\frac{\rho^4 z(l_2^2+\rho^2\bar{t})(l_2^2-a^2)^{1/2}}{(l_2^2-\rho^2)(l_2^2-\rho^2\bar{t})^2(l_2^2-l_1^2)}\right\}, \quad (55)$$

$$g_{15}(z) = \frac{\rho^2 e^{2i\phi}(\rho_0^2-a^2)^{1/2}(a^2-l_1^2)^{1/2}\bar{t}^2(a^2+l_1^2\bar{t})}{a^2(l_2^2-l_1^2)(a^2-l_1^2\bar{t})^2}, \quad (56)$$

$$g_{16}(z) = \frac{1}{\bar{q}}\left\{\frac{R_0^2+z^2}{\bar{q}R_0}\tan^{-1}\left(\frac{j}{R_0}\right)-\frac{2z}{\bar{q}}\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right)\right.$$

$$-(\rho_0^2-a^2)^{1/2}\left[\frac{z}{\bar{s}}\left(\frac{2}{\bar{q}}+\frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}\right)\left(\tan^{-1}\left(\frac{\bar{s}}{(a^2-l_1^2)^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{s}}{a}\right)\right)\right.$$

$$\left.\left.-\frac{e^{i\phi_0}}{\rho_0(1-\bar{\zeta})^{1/2}}\tan^{-1}\left(\frac{a(1-\bar{\zeta})^{1/2}}{(l_2^2-a^2)^{1/2}}\right)\right]+\frac{ja e^{i\phi}}{\rho\bar{s}^2}(a-(a^2-l_1^2)^{1/2})\right\}. \quad (57)$$

It should be remembered that the notations $\zeta, q, R, t, s, R_0, j$ are defined in (15), (17), (19) and (A29), respectively.

We should also notice the identities

$$\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}} = \frac{a}{\bar{s}}, \quad \frac{\bar{t}^{1/2}(a^2-l_1^2)^{1/2}}{a(1-\bar{t})^{1/2}} = \frac{(a^2-l_1^2)^{1/2}}{s}. \quad (58)$$

and this means that the trigonometric functions which were introduced in various formulae in different manner, are in fact the same, for example

$$\tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-l_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right)=\tan^{-1}\left(\frac{s}{(a^2-l_1^2)^{1/2}}\right)-\tan^{-1}\left(\frac{s}{a}\right), \quad (59)$$

and yet another example:

$$\tan^{-1}\left(\frac{(a^2-\rho^2\bar{t})^{1/2}}{(l_2^2-a^2)^{1/2}}\right)=\tan^{-1}\left(\frac{a(1-\bar{\zeta})^{1/2}}{(l_2^2-a^2)^{1/2}}\right). \quad (60)$$

Every function g_i depends on the coordinate of the field point (ρ, ϕ, z) and the coordinates $(\rho_0, \phi_0, 0)$ of the point of application of the force T. We use notation $g_i(z)$ just to emphasize the fact that one should substitute z_k ($k = 1, 2, 3$) instead of z when using formulae (37)–(42). The reader can also notice the absence of $g_6(z)$ from the list above. This was not an oversight, but rather to preserve in formulae (37)–(42) the form of solution used in Fabrikant (1989) for a penny-shaped crack, where the equivalent notation f_6 was used elsewhere.

The expressions (37)–(42) simplify significantly on the plane $z = 0$. The results are:

$$u = 0, \quad \text{for } 0 \leqslant \rho \leqslant a,$$

$$u = \frac{G_1}{\pi}\left[\frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right)-\frac{G_2^2}{G_1^2}\frac{t^2(1+t)\eta}{a^2(1-t)^2}\right]T$$
$$+\frac{G_2}{\pi}\left[\frac{q}{\bar{q}R}\tan^{-1}\left(\frac{\eta}{R}\right)+\frac{\eta}{\bar{q}}\left(\frac{t\,e^{i\phi}}{\rho(1-t)}-\frac{\bar{t}\,e^{i\phi_0}}{\rho_0(1-\bar{t})}\right)\right]\bar{T} \quad \text{for } \rho > a, \quad (61)$$

$$w = \frac{2}{\pi}H\alpha\,\mathrm{Re}\left\{\left\{\frac{1}{q}\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right)+\frac{(\rho_0^2-a^2)^{1/2}}{qs}\left(\tan^{-1}\left(\frac{s}{(a^2-\rho^2)^{1/2}}\right)\right.\right.\right.$$
$$\left.-\tan^{-1}\left(\frac{s}{a}\right)\right)+\frac{G_2}{G_1}\frac{a(\rho_0^2-a^2)^{1/2}}{\rho_0\,e^{i\phi_0}}\left[\frac{a}{\bar{s}^3}\left(\tan^{-1}\left(\frac{\bar{s}}{(a^2-l_1^2)^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{s}}{a}\right)\right)\right.$$
$$\left.\left.\left.-\frac{1}{\bar{s}^2}\left(1-\frac{(a^2-\rho^2)^{1/2}}{a(1-\zeta)}\right)\right]\right\}T\right\}, \quad \text{for } \rho < a,$$

$$w = \frac{2}{\pi}H\alpha\,\mathrm{Re}\left\{\left[\frac{1}{q}\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right)+\frac{(\rho_0^2-a^2)^{1/2}}{qs}\tan^{-1}\left(\frac{a}{s}\right)\right.\right.$$
$$\left.\left.+\frac{G_2}{G_1}\frac{a(\rho_0^2-a^2)^{1/2}}{\rho_0\,e^{i\phi_0}}\left(\frac{a}{\bar{s}^3}\tan^{-1}\left(\frac{a}{s}\right)-\frac{1}{\bar{s}^2}\right)\right]T\right\}, \quad \text{for } \rho > a, \quad (62)$$

$$\sigma_1 = 0, \quad \text{for } \rho < a,$$

$$\sigma_1 = \frac{2}{\pi^2}\mathrm{Re}\left\{\left[\left(2\pi HA_{66}\gamma_1\gamma_2-\frac{\gamma_1+\gamma_2}{\gamma_1\gamma_2}\right)\left(\frac{1}{qR}\tan^{-1}\left(\frac{\eta}{R}\right)-\frac{\eta a^2}{s^2\bar{s}^2}\left(\frac{\rho\,e^{-i\phi}}{\rho^2-a^2}-\frac{1}{q}\right)\right)\right.\right.$$
$$\left.\left.-\frac{G_2}{G_1}\frac{\eta\rho\,e^{-i\phi}t^2(1+t)}{a^2(1-t)^2(\rho^2-a^2)}\right]T\right\}, \quad \text{for } \rho > a, \quad (63)$$

$$\sigma_2 = \frac{1}{\pi^2}\left\{\left(2\pi A_{66}H\gamma_1\gamma_2+\frac{1}{\gamma_3}\right)\left[g_5(0)T+\frac{G_2}{G_1}g_{12}(0)\bar{T}\right]\right.$$
$$\left.+\left(2\pi A_{66}H\gamma_1\gamma_2-\frac{1}{\gamma_3}\right)\left[g_{11}(0)\bar{T}+\frac{G_2}{G_1}\bar{g}_{13}(0)T\right]\right\}, \quad (64)$$

$$\sigma_z = 0, \tag{65}$$

$$\tau_z = \frac{1}{\pi^2}\frac{(\rho_0^2-a^2)^{1/2}}{(a^2-\rho^2)^{1/2}}\left[\frac{T}{R^2}+\frac{G_2}{G_1}\frac{\rho^2\,e^{2i\phi}\bar{t}^2(a^2+\rho^2\bar{t})}{a^2(a^2-\rho^2\bar{t})^2}\bar{T}\right]$$

$$= \frac{1}{\pi^2}\frac{(\rho_0^2-a^2)^{1/2}}{(a^2-\rho^2)^{1/2}}\left[\frac{T}{R^2}+\frac{G_2}{G_1}\frac{e^{2i\phi_0}(1+\bar{\zeta})}{\rho_0^2(1-\bar{\zeta})^2}\bar{T}\right], \quad \text{for } \rho < a,$$

$$\tau_z = -T\delta(\rho-\rho_0)\delta(\phi-\phi_0), \quad \text{for } \rho > a. \tag{66}$$

The second and third mode stress intensity factors can be expressed through the decomposition $\tau^{(n)} = \tau_{zn}+i\tau_{tz}$, which is related to τ_z by a relationship $\tau^{(n)} = \tau_z e^{-i\phi}$. Introducing the complex stress intensity factor

$$k_2+ik_3 = \lim_{\rho\to a}[(a-\rho)^{1/2}\tau_z e^{-i\phi}]. \tag{67}$$

From (66)

$$k_2+ik_3 = \frac{(\rho_0^2-a^2)^{1/2}}{\pi^2(2a)^{1/2}}\left[\frac{Te^{-i\phi}}{\rho_0^2+a^2-2a\rho_0\cos(\phi-\phi_0)} + \frac{G_2}{G_1}\frac{e^{-i(\phi-\phi_0)}(\rho_0 e^{-i\phi_0}+a e^{-i\phi})}{\rho_0(\rho_0 e^{-i\phi_0}-a e^{-i\phi})^2}\bar{T}\right]. \tag{68}$$

In the case of a distributed loading, the stress intensity factors are given by

$$k_2+ik_3 = \frac{e^{-i\phi}}{\pi^2(2a)^{1/2}}\left\{\int_0^{2\pi}\int_a^\infty \frac{(\rho_0^2-a^2)^{1/2}\tau(\rho_0,\phi_0)\rho_0\,d\rho_0\,d\phi_0}{\rho_0^2+a^2-2a\rho_0\cos(\phi-\phi_0)} + \frac{G_2}{G_1}\int_0^{2\pi}\int_a^\infty \frac{(\rho_0^2-a^2)^{1/2}(\rho_0+a e^{-i(\phi-\phi_0)})\bar{\tau}(\rho_0,\phi_0)\rho_0\,d\rho_0\,d\phi_0}{\rho_0(\rho_0 e^{-i\phi_0}-a e^{-i\phi})^2}\right\}. \tag{69}$$

Note that eqn (69) is in agreement with (14).

Formulae (37)–(57) are the main new results of this paper.

DISCUSSION

The complete solution, obtained in the previous section, is of great value because it allows one to solve easily many complicated problems which were not even attempted before. I consider, as an example, interaction between an arbitrarily located horizontal force Q and an external circular crack of radius a: Let $Q = Q_x+iQ_y$, where Q_x and Q_y are the x and y components of Q. Let the force Q be applied at an arbitrary point $M(\rho,\phi,z)$. We need to find the stress intensity factor at the crack boundary.

The solution can be obtained in an elementary way by using the reciprocal theorem. We consider two systems in equilibrium. The first one is an elastic space weakened by an external circular crack, with two equal and oppositely directed unit tangential forces $T = 1+i$ applied at the points $(\rho_0, \phi_0, 0^\pm)$ of crack faces. The second system is the same space, with the crack faces tractions free, and the horizontal force $Q = Q_x+iQ_y$ applied at the point $M(\rho,\phi,z)$. For simplicity of the transformation to follow, we present eqn (37) in a generalized form

$$u = D_1 T + D_2 \bar{T}. \tag{70}$$

Here D_1 and D_2 are the combined factors of T and $\bar{T}$, respectively. The tangential displacements at the point $M(\rho, \phi, z)$ in the x and y directions owing to a couple of unit forces T_x will be, respectively,

$$(u_x)_{T_x} = \mathrm{Re}\,(D_1 + D_2), \quad (u_y)_{T_x} = \mathrm{Im}\,(D_1 + D_2). \tag{71}$$

The similar displacements due to a pair of unit forces T_y are

$$(u_x)_{T_y} = \mathrm{Re}\,[(D_1 - D_2)\mathrm{i}] = -\mathrm{Im}\,(D_1 - D_2), \quad (u_y)_{T_y} = \mathrm{Im}\,[(D_1 - D_2)\mathrm{i}] = \mathrm{Re}\,(D_1 - D_2). \tag{72}$$

Denoting the tangential displacement discontinuity in the x direction due to force Q_x as $(\Delta_x)_{Q_x}$, according to the reciprocal theorem, we have

$$(\Delta_x)_{Q_x} = Q_x \,\mathrm{Re}\,(D_1 + D_2). \tag{73}$$

The remaining three equations are obtained in a similar manner, and they are

$$(\Delta_y)_{Q_x} = -Q_x \,\mathrm{Im}\,(D_1 - D_2), \quad (\Delta_x)_{Q_y} = Q_y \,\mathrm{Im}\,(D_1 + D_2), \quad (\Delta_y)_{Q_y} = Q_y \,\mathrm{Re}\,(D_1 - D_2). \tag{74}$$

The meaning of the notation in eqns (74) is the same as in eqns (71)–(72). Summation of eqn (73) with the first expression of (74) multiplied by i yields

$$(\Delta)_{Q_x} = (\Delta_x)_{Q_x} + \mathrm{i}(\Delta_y)_{Q_x} = Q_x[\bar{D}_1 + D_2]. \tag{75}$$

A similar operation with the second and the third expressions of (74) gives

$$(\Delta)_{Q_y} = (\Delta_x)_{Q_y} + \mathrm{i}(\Delta_y)_{Q_y} = Q_y[\mathrm{i}\bar{D}_1 - \mathrm{i}D_2]. \tag{76}$$

Finally, summation of eqns (75) and (76) results in

$$\Delta_Q = (\Delta)_{Q_x} + (\Delta)_{Q_y} = \bar{D}_1 Q + D_2 \bar{Q}. \tag{77}$$

A comparison of eqns (70) and (77) shows how the reciprocal theorem can be used in the case of complex forces and displacements: we can obtain the tangential displacement discontinuity at the point $(\rho_0, \phi_0, 0)$ owing to a tangential force Q applied at the point (ρ, ϕ, z) by using the expression for tangential displacements at the point (ρ, ϕ, z) owing to a pair of equal and oppositely directed tangential forces T applied at the point $(\rho_0, \phi_0, 0^{\pm})$, by way of substituting Q instead of T, and by replacing the coefficient of Q by its complex conjugate. Using this rule, we have from eqn (37)

$$\Delta_Q = \frac{G_1 - G_2}{2\pi} \sum_{k=1}^{2} \frac{1}{m_k - 1} \left\{ -\left[g_2(z_k) + \frac{G_2}{G_1} g_7(z_k) \right] Q + \left[g_{16}(z_k) + \frac{G_2}{G_1} g_8(z_k) \right] \bar{Q} \right\}$$
$$+ \frac{G_1 + G_2}{2\pi} \left\{ \left[g_2(z_3) - \frac{G_2}{G_1} g_7(z_3) \right] Q + \left[g_{16}(z_3) - \frac{G_2}{G_1} g_8(z_3) \right] \bar{Q} \right\}. \tag{78}$$

The stress intensity factors of the second and third kind can be expressed through the tangential displacement discontinuity (Fabrikant, 1989) as

$$k_2+\mathrm{i}k_3 = -\frac{a}{2\pi(G_1^2-G_2^2)\sqrt{2a}}\lim_{\rho_0\to a}\left[\frac{G_1\,\mathrm{e}^{-\mathrm{i}\phi_0}\Delta_Q+G_2\,\mathrm{e}^{\mathrm{i}\phi_0}\bar{\Delta}_Q}{(\rho_0^2-a^2)^{1/2}}\right]. \tag{79}$$

The limiting quantities, which need to be computed, are as follows

$$\lim_{\rho_0\to a}\left\{\frac{g_2(z)}{(\rho_0^2-a^2)^{1/2}}\right\} = \frac{(l_2^2-a^2)^{1/2}}{aR_a^2},$$

$$\lim_{\rho_0\to a}\left\{\frac{g_7(z)}{(\rho_0^2-a^2)^{1/2}}\right\} = \frac{z}{a^3}\left\{\frac{3\bar{t}_a^{5/2}}{(1-\bar{t}_a)^{5/2}}\left[\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{s_a}\right)-\tan^{-1}\left(\frac{a}{s_a}\right)\right]\right.$$

$$\left.+\frac{\bar{t}_a^2}{(1-\bar{t}_a)^2}\left[\frac{a(a^2-l_1^2)^{1/2}}{a^2-l_1^2\bar{t}_a}-2-\bar{t}_a+\frac{a(1+\bar{t}_a)}{(a^2-l_1^2)^{1/2}}\right]\right\},$$

$$\lim_{\rho_0\to a}\left\{\frac{g_8(z)}{(\rho_0^2-a^2)^{1/2}}\right\} = \mathrm{e}^{2\mathrm{i}\phi_0}\left[\frac{a}{(a^2-\rho^2\bar{t}_a)^{3/2}}\tan^{-1}\left(\frac{(a^2-\rho^2\bar{t}_a)^{1/2}}{(l_2^2-a^2)^{1/2}}\right)-\frac{\bar{t}_a l_1(\rho^2-l_1^2)^{1/2}}{(a^2-l_1^2\bar{t}_a)(a^2-\rho^2\bar{t}_a)}\right],$$

$$\lim_{\rho_0\to a}\left\{\frac{g_{16}(z)}{(\rho_0^2-a^2)^{1/2}}\right\} = \frac{1}{\bar{q}_a}\left\{\frac{(R_a^2+z^2)(l_2^2-a^2)^{1/2}}{\bar{q}_a R_a^2}-\frac{2z}{\bar{q}_a a}\right.$$

$$+\left[\frac{\mathrm{e}^{\mathrm{i}\phi_0}}{(a^2-\rho^2\bar{t}_a)^{1/2}}\tan^{-1}\left(\frac{(a^2-\rho^2\bar{t}_a)^{1/2}}{(l_2^2-a^2)^{1/2}}\right)-\frac{z}{\bar{s}_a}\left(\frac{2}{\bar{q}_a}+\frac{a\,\mathrm{e}^{\mathrm{i}\phi_0}}{\bar{s}_a^2}\right)\right]\left[\tan^{-1}\left(\frac{a}{\bar{s}_a}\right)\right.$$

$$\left.\left.-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{\bar{s}_a}\right)\right]+\frac{(l_2^2-a^2)^{1/2}\mathrm{e}^{\mathrm{i}\phi}}{\rho\bar{s}_a^2}(a-(a^2-l_1^2)^{1/2})\right\}. \tag{80}$$

Here the following notation is introduced

$$R_a^2 = \rho^2+a^2-2\rho a\cos(\phi-\phi_0)+z^2,\quad s_a = \sqrt{\rho a\,\mathrm{e}^{\mathrm{i}(\phi-\phi_0)}-a^2},$$

$$t_a = \frac{a}{\rho}\mathrm{e}^{\mathrm{i}(\phi-\phi_0)},\quad q_a = \rho\,\mathrm{e}^{\mathrm{i}\phi}-a\,\mathrm{e}^{\mathrm{i}\phi_0}. \tag{81}$$

The overbar everywhere denotes the complex conjugate quantity. Substitution of eqns (80) and (78) in (79) gives the required stress intensity factors. It would be too cumbersome to write the final expression explicitly. A significant simplification takes place when $z = 0$. We can obtain from eqn (61)

$$\Delta_Q = \frac{G_1}{\pi}\left[\frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right)-\frac{G_2^2}{G_1^2}\frac{\bar{t}^2(1+\bar{t})\eta}{a^2(1-\bar{t})^2}\right]Q$$

$$+\frac{G_2}{\pi}\left[\frac{q}{\bar{q}R}\tan^{-1}\left(\frac{\eta}{R}\right)+\frac{\eta}{\bar{q}}\left(\frac{t\,\mathrm{e}^{\mathrm{i}\phi}}{\rho(1-t)}-\frac{\bar{t}\,\mathrm{e}^{\mathrm{i}\phi_0}}{\rho_0(1-\bar{t})}\right)\right]\bar{Q}. \tag{82}$$

The limit can be computed easily

$$\lim_{\rho_0\to a}\left\{\frac{\Delta_Q}{(\rho_0^2-a^2)^{1/2}}\right\} = \frac{G_1}{\pi}\frac{(\rho^2-a^2)^{1/2}}{a}\left[\frac{1}{R_A^2}-\frac{G_2^2}{G_1^2}\frac{\bar{t}_a^2(1+\bar{t}_a)}{a^2(1-\bar{t}_a)^2}\right]Q$$

$$+\frac{G_2}{\pi}\frac{(\rho^2-a^2)^{1/2}}{a\bar{q}_a}\left[\frac{1}{\bar{q}_a}+\frac{t_a}{\bar{q}_a}-\frac{\bar{t}_a\,\mathrm{e}^{\mathrm{i}\phi_0}}{a(1-\bar{t}_a)}\right]\bar{Q}, \tag{83}$$

and its substitution in eqn (79) yields

$$k_2+\mathrm{i}k_3 = \frac{\mathrm{e}^{-\mathrm{i}\phi_0}}{2\pi^2\sqrt{2a}}\left[\frac{Q}{R_A^2}+\frac{G_2}{G_1}\frac{\mathrm{e}^{\mathrm{i}\phi}(\rho\,\mathrm{e}^{-\mathrm{i}\phi}+a\,\mathrm{e}^{-\mathrm{i}\phi_0})}{\rho(\rho\,\mathrm{e}^{-\mathrm{i}\phi}-a\,\mathrm{e}^{-\mathrm{i}\phi_0})^2}\bar{Q}\right], \tag{84}$$

with $R_A^2 = \rho^2+a^2-2\rho a\cos(\phi-\phi_0)$. The result (84) corresponds to half of the expression (68), as it should be, since there is one-sided loading of the crack.

The presented complete solution to the external circular crack problem provides a powerful basis for solving various difficult problems of interaction of arbitrary located forces with the crack, interaction between cracks, etc.

REFERENCES

Elliott, H. A. (1948). Three-dimensional stress distribution in hexagonal aeolotropic crystals. *Proc. Cambridge Phil. Soc.* **44**, 522–533.

Fabrikant, V. I. (1989). *Application of Potential Theory in Mechanics. Selection of New Results.* Kluwer, Dordrecht.

APPENDIX A

Computation of various derivatives of L_1 and L_2 as they are defined in eqn (32) are presented here. The simplest integral to compute is

$$I_1 = \iint_S \frac{B_2(N,N_0)}{R(M,N)}\,\mathrm{d}S_N$$

$$= \int_0^{2\pi}\int_a^\infty \frac{a\,\mathrm{e}^{-\mathrm{i}(\psi-\phi_0)}(r\rho_0\,\mathrm{e}^{\mathrm{i}(\psi-\phi_0)}+a^2)}{r\rho_0(r\rho_0\,\mathrm{e}^{\mathrm{i}(\psi-\phi_0)}-a^2)^2}\,\frac{\sqrt{r^2-a^2}\sqrt{\rho_0^2-a^2}\,r\,\mathrm{d}r\,\mathrm{d}\psi}{[r^2+\rho^2-2r\rho\cos(\phi-\psi)+z^2]^{1/2}}. \tag{A1}$$

We use the integral representation from Fabrikant (1989)

$$\frac{1}{R(M,N)} = \frac{1}{[r^2+\rho^2-2r\rho\cos(\phi-\psi)+z^2]^{1/2}} = \frac{2}{\pi}\int_{l_2(r)}^\infty \frac{\lambda(\rho r/x^2,\phi-\psi)\,\mathrm{d}x}{[(x^2-\rho^2)(g^2(x)-r^2)]^{1/2}}. \tag{A2}$$

Here

$$\lambda(k,\vartheta) = \frac{1-k^2}{1+k^2-2k\cos\vartheta} = \sum_{n=-\infty}^{\infty} k^{|n|}\,\mathrm{e}^{\mathrm{i}n\vartheta}, \quad \text{for } k<1. \tag{A3}$$

$$l_2(r) = \tfrac{1}{2}[\sqrt{(r+\rho)^2+z^2}+\sqrt{(r-\rho)^2+z^2}], \quad g(x) = x\left(1-\frac{z^2}{x^2-\rho^2}\right)^{1/2}. \tag{A4}$$

The function g is inverse to l_2, so that $g[l_2(r)] = r$. I_1 is transformed by substituting eqn (A2) in (A1) and expanding B_2 in the Fourier series:

$$I_1 = \int_0^{2\pi}\int_a^\infty \frac{\sqrt{r^2-a^2}\sqrt{\rho_0^2-a^2}}{a^3}\left[\left(\frac{a^2\,\mathrm{e}^{-\mathrm{i}(\psi-\phi_0)}}{r\rho_0}\right)^2\sum_{n=0}^\infty(2n+1)\left(\frac{a^2\,\mathrm{e}^{-\mathrm{i}(\psi-\phi_0)}}{r\rho_0}\right)^n\right]$$

$$\times\left[\frac{2}{\pi}\int_{l_2(r)}^\infty \frac{\lambda(\rho r/x^2,\phi-\psi)\,\mathrm{d}x}{[(x^2-\rho^2)(g^2(x)-r^2)]^{1/2}}\right]r\,\mathrm{d}r\,\mathrm{d}\psi$$

$$= 4\frac{\sqrt{\rho_0^2-a^2}}{a^3}\int_{l_2}^\infty \frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}}\left[\int_a^{g(x)}\frac{\sqrt{r^2-a^2}\,r\,\mathrm{d}r}{\sqrt{g^2(x)-r^2}}\right]\sum_{n=0}^\infty(2n+1)\left(\frac{a^2\rho}{x^2\rho_0}\mathrm{e}^{-\mathrm{i}(\phi-\phi_0)}\right)^{n+2}$$

$$= \pi\frac{\sqrt{\rho_0^2-a^2}}{a^3}\int_{l_2}^\infty \frac{g^2(x)-a^2}{\sqrt{x^2-\rho^2}}\left(\frac{a^2\rho}{x^2\rho_0}\mathrm{e}^{-\mathrm{i}(\phi-\phi_0)}\right)^2\frac{1+\dfrac{a^2\rho}{x^2\rho_0}\mathrm{e}^{-\mathrm{i}(\phi-\phi_0)}}{\left(1-\dfrac{a^2\rho}{x^2\rho_0}\mathrm{e}^{-\mathrm{i}(\phi-\phi_0)}\right)^2}\mathrm{d}x$$

$$= \pi\bar{\varepsilon}^4\frac{\sqrt{\rho_0^2-a^2}}{a^3}\int_{l_2}^\infty \frac{(x^2-l_1^2)(x^2-l_2^2)(x^2+\bar{\varepsilon}^2)}{x^2(x^2-\bar{\varepsilon}^2)^2(x^2-\rho^2)^{3/2}}\,\mathrm{d}x \tag{A5}$$

The abbreviation l_2 stands for $l_2(a)$, as it is defined in eqn (A4), and

$$l_1 \equiv l_1(a) = \tfrac{1}{2}[\sqrt{(a+\rho)^2+z^2} - \sqrt{(a-\rho)^2+z^2}], \quad \bar{\varepsilon}^2 = a^2 \frac{\rho}{\rho_0} e^{i(\phi-\phi_0)}. \tag{A6}$$

The following rule of interchange of the order of integration was used in eqn (A5)

$$\int_a^l dr \int_{l_2(r)}^l dx = \int_{l_2}^{\infty} dx \int_a^{g(x)} dr. \tag{A7}$$

Now computation of I_1 has been reduced to an elementary single integral. The integrand in eqn (A5) can be decomposed in simple fractions

$$\frac{(x^2-l_1^2)(x^2-l_2^2)(x^2+\bar{\varepsilon}^2)}{x^2(x^2-\bar{\varepsilon}^2)^2} = 1 + \frac{a^2\rho^2}{\bar{\varepsilon}^2 x^2} + \frac{3\bar{\varepsilon}^4-(l_1^2+l_2^2)\bar{\varepsilon}^2-a^2\rho^2}{\bar{\varepsilon}^2(x^2-\bar{\varepsilon}^2)} + \frac{2(\bar{\varepsilon}^2-l_1^2)(\bar{\varepsilon}^2-l_2^2)}{(x^2-\bar{\varepsilon}^2)^2}. \tag{A8}$$

Substitution of eqn (A8) in (A5) allows one finally to compute

$$\begin{aligned} I_1 = \pi \frac{\sqrt{\rho_0^2-a^2}}{a^3}\bar{t}^2 \Bigg\{ & \frac{z^2(1+\bar{t})}{(1-\bar{t})^2}\left(1-\frac{a}{(a^2-l_1^2)^{1/2}}\right) - \frac{a^2}{\bar{t}}\left(1-\frac{(a^2-l_1^2)^{1/2}}{a}\right) \\ & + \frac{\rho^2\bar{t}^2-\bar{t}(a^2+\rho^2+z^2)+a^2}{\bar{t}(1-\bar{t})^2}\left[\frac{a(a^2-l_1^2)^{1/2}}{a^2-\bar{t}l_1^2}-1\right] + \frac{1}{\bar{t}^{1/2}}\left[\frac{a^2-\rho^2\bar{t}}{(1-\bar{t})^{3/2}} - \frac{2a^2}{\bar{t}(1-\bar{t})^{1/2}}\right. \\ & \left. - \frac{3z^2\bar{t}}{(1-\bar{t})^{5/2}}\right]\left[\tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-l_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right) - \tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right)\right]\Bigg\}, \end{aligned} \tag{A9}$$

where t is defined in eqn (19).

The next integral to be computed is

$$I_2 = \int_S\int \frac{zB_2(N,N_0)}{R^3(M,N)} dS_N = -\frac{\partial I_1}{\partial z}. \tag{A10}$$

Differentiating eqn (A5) with respect to z, we obtain

$$\begin{aligned} I_2 &= 2\pi z\bar{\varepsilon}^4 \frac{\sqrt{\rho_0^2-a^2}}{a^3}\int_{l_2}^{l} \frac{(x^2+\bar{\varepsilon}^2)\,dx}{(x^2-\bar{\varepsilon}^2)^2(x^2-\rho^2)^{3/2}} \\ &= 2\pi z \frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{\frac{3\bar{t}^{5/2}}{(1-\bar{t})^{5/2}}\left[\tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-l_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right) - \tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right)\right]\right. \\ &\quad \left. + \frac{\bar{t}^2}{(1-\bar{t})^2}\left[\frac{a(a^2-l_1^2)^{1/2}}{a^2-l_1^2\bar{t}} - 2 - \bar{t} + \frac{a(1+\bar{t})}{(a^2-l_1^2)^{1/2}}\right]\right\}. \end{aligned} \tag{A11}$$

Application of the operator Λ to the complex conjugate of eqn (A5) yields

$$\begin{aligned} I_3 = \Lambda\bar{I}_1 &= \pi\varepsilon^4 \frac{\sqrt{\rho_0^2-a^2}}{a^3}\rho\, e^{i\phi}\left\{3\int_{l_2}^{l} \frac{(x^2-l_1^2)(x^2-l_2^2)(x^2+\varepsilon^2)}{x^2(x^2-\varepsilon^2)^2(x^2-\rho^2)^{5/2}}dx\right. \\ &\quad \left. -2\int_{l_2}^{l} \frac{(x^2-a^2)(x^2+\varepsilon^2)\,dx}{x^2(x^2-\varepsilon^2)^2(x^2-\rho^2)^{3/2}}\right\} = \pi\varepsilon^4 \frac{\sqrt{\rho_0^2-a^2}}{a^3}\rho\, e^{i\phi}\left\{\int_{l_2}^{\infty} \frac{(x^2-l_1^2)(x^2-l_2^2)(x^2+\varepsilon^2)}{x^2(x^2-\varepsilon^2)^2(x^2-\rho^2)^{5/2}}dx\right. \\ &\quad \left. -2z^2\int_{l_2}^{l} \frac{(x^2+\varepsilon^2)\,dx}{(x^2-\varepsilon^2)^2(x^2-\rho^2)^{5/2}}\right\} = \pi\rho\, e^{i\phi}\frac{\sqrt{\rho_0^2-a^2}}{a^3}t^2\left\{\left(1-\frac{a}{(a^2-l_1^2)^{1/2}}\right)\left[\frac{a^2}{\rho^2 t}\right.\right. \\ &\quad \left. + \frac{(1+t)(a^2-2z^2-\rho^2)}{(1-t)^2\rho^2} + \frac{1}{(1-t)^2\rho^2}\left(9z^2-\frac{15z^2}{1-t}+\frac{a^2-\rho^2 t}{t}\right)\right] \\ &\quad - \frac{a}{(a^2-l_1^2)^{1/2}}\left[\frac{1+t}{(1-t)^2}\left(1-\frac{a^2}{l_2^2}\right) - \frac{a^2}{l_2^2 t} + \frac{3z^2}{(1-t)^2(l_2^2-\rho^2 t)} - \frac{a^2-\rho^2 t}{t(1-t)(l_2^2-\rho^2 t)}\right] \\ &\quad + \frac{1}{\rho^2\sqrt{t}}\left[\frac{2a^2}{t(1-t)^{3/2}} - \frac{3(a^2-\rho^2 t)}{(1-t)^{5/2}} + \frac{15z^2 t}{(1-t)^{7/2}}\right]\left[\tan^{-1}\left(\frac{t^{1/2}(a^2-l_1^2)^{1/2}}{a(1-t)^{1/2}}\right)\right. \\ &\quad \left.\left. - \tan^{-1}\left(\frac{t^{1/2}}{(1-t)^{1/2}}\right)\right]\right\}. \end{aligned} \tag{A12}$$

By applying yet another Λ-operator to eqn (A12), we have

$$I_4 = \Lambda^2 \bar{I}_1 = \pi\varepsilon^4 \frac{\sqrt{\rho_0^2-a^2}}{a^3}\rho\, e^{i\phi}\left\{\Lambda \int_{l_2}^{\infty} \frac{(x^2-l_1^2)(x^2-l_2^2)(x^2+\varepsilon^2)}{x^2(x^2-\varepsilon^2)^2(x^2-\rho^2)^{5/2}}\,dx\right.$$

$$\left. -2z^2\Lambda \int_{l_2}^{\infty} \frac{(x^2+\varepsilon^2)\,dx}{(x^2-\varepsilon^2)^2(x^2-\rho^2)^{5/2}}\right\}$$

$$= \pi\varepsilon^4\rho^2 e^{2i\phi}\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{3\int_{l_2}^{\infty} \frac{(x^2-a^2)(x^2+\varepsilon^2)\,dx}{x^2(x^2-\varepsilon^2)^2(x^2-\rho^2)^{5/2}}\right.$$

$$\left. -15z^2\int_{l_2}^{\infty} \frac{(x^2+\varepsilon^2)\,dx}{(x^2-\varepsilon^2)^2(x^2-\rho^2)^{7/2}} + \frac{2z^2(l_2^2+\varepsilon^2)(l_2^2-a^2)}{l_2(l_2^2-\varepsilon^2)^2(l_2^2-\rho^2)^{5/2}(l_2^2-l_1^2)}\right\}. \qquad \text{(A13)}$$

The integrals in eqn (A13) are elementary, we can use the indefinite integrals presented in Appendix B for their evaluation. The final result is

$$I_4 = \Lambda^2 \iint_S \frac{\bar{B}_2(N,N_0)}{R(M,N)}\,dS_N = \pi\rho^2 e^{2i\phi}\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{\frac{2\rho^4 t^2(l_2^2+\rho^2 t)(l_2^2-a^2)^2}{l_2^3(l_2^2-\rho^2 t)^2(l_2^2-\rho^2)^{3/2}(l_2^2-l_1^2)}\right.$$

$$-\frac{l_2^3 t^2(1+t)(2\rho^2-3l_1^2+a^2)}{\rho^2(1-t)^2(l_2^2-\rho^2)^{3/2}} + t^2\left(1-\frac{a}{(a^2-l_1^2)^{1/2}}\right)\left[\frac{8+9t-2t^2}{\rho^2(1-t)^3}\right.$$

$$\left. + \frac{a^2(-6+t-18t^2+8t^3)}{\rho^4 t(1-t)^3} + \frac{z^2(48+87t-38t^2+8t^3)}{\rho^4(1-t)^4}\right] - \frac{3al_1^2 t}{\rho^3(a^2-l_1^2)^{1/2}}$$

$$-\frac{15t^2z^2l_2^3(a^2-l_1^2)^{1/2}}{a(1-t)^4\rho^4(l_2^2-\rho^2 t)} - \frac{at}{(a^2-l_1^2)^{1/2}}\left[\frac{3(a^2-\rho^2 t)}{\rho^2(1-t)^2(l_2^2-\rho^2 t)}\right.$$

$$\left. -tz^2\left(\frac{15}{\rho^4(1-t)^4} + \frac{9+15t-4t^2}{\rho^2(1-t)^3(l_2^2-\rho^2)}\right)\right] + \frac{t^{3/2}}{\rho^4(1-t)^{1/2}}\left[\frac{6a^2}{t(1-t)^2}\right.$$

$$\left. \left. -\frac{15(a^2-\rho^2 t)}{(1-t)^3} + \frac{105tz^2}{(1-t)^4}\right]\left[\tan^{-1}\left(\frac{t^{1/2}}{(1-t)^{1/2}}\right) - \tan^{-1}\left(\frac{t^{1/2}(a^2-l_1^2)^{1/2}}{a(1-t)^{1/2}}\right)\right]\right\}. \qquad \text{(A14)}$$

Integration with respect to z of eqn (A1) gives

$$L_2(M,N_0) = \iint_S B_2(N,N_0)\ln[R(M,N)+z]\,dS_N = \pi\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{\frac{(\bar{t}+2)[z^3-(l_2^2-a^2)^{3/2}]\bar{t}^2}{3(1-\bar{t})^2}\right.$$

$$+\bar{t}al_1(\rho^2-l_1^2)^{1/2}\left[\frac{\bar{t}(\rho^2-l_1^2)}{l_1^2(1-\bar{t})^2}-1\right] + \bar{t}((l_2^2-a^2)^{1/2}-z)\left(\frac{a^2-\rho^2\bar{t}}{1-\bar{t}}+a^2\right)$$

$$+a(\rho^2\bar{t}-2a^2)\sin^{-1}\left(\frac{a}{l_2}\right) + 2a^2(a^2-\rho^2\bar{t})^{1/2}\sin^{-1}\left(\frac{(a^2-\rho^2\bar{t})^{1/2}}{(l_2^2-\rho^2\bar{t})^{1/2}}\right)$$

$$+\frac{z\bar{t}^{3/2}}{(1-\bar{t})^{1/2}}\left(\frac{a^2-\rho^2\bar{t}}{1-\bar{t}} - \frac{2a^2}{\bar{t}} - \frac{z^2\bar{t}}{(1-\bar{t})^2}\right)\left[\tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-l_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right)\right.$$

$$\left.\left. -\tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right)\right]\right\}. \qquad \text{(A15)}$$

Indefinite integrals from Appendix B were used here. Application of the Λ-operator to the complex conjugate of eqn (A11) yields

$$\Lambda\frac{\partial}{\partial z}\iint_S \frac{\bar{B}_2(N,N_0)}{R(M,N)}\,dS_N = 2\pi z t^2\rho\, e^{i\phi}\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{\frac{\rho^4(l_2^2+\rho^2 t)(l_2^2-a^2)}{l_2(l_2^2-\rho^2 t)^2(l_2^2-\rho^2)^{3/2}(l_2^2-l_1^2)}\right.$$

$$-\frac{15t^{1/2}}{\rho^2(1-t)^{7/2}}\left[\tan^{-1}\left(\frac{t^{1/2}}{(1-t)^{1/2}}\right) - \tan^{-1}\left(\frac{t^{1/2}(a^2-l_1^2)^{1/2}}{a(1-t)^{1/2}}\right)\right]$$

$$-\frac{1}{(1-t)^2}\left[\frac{2(1+t)}{\rho^2}+\frac{6+9t}{\rho^2(1-t)}\right] + \frac{a}{(a^2-l_1^2)^{1/2}(1-t)^2}\left[\frac{2(1+t)}{\rho^2}\right.$$

$$\left.\left. +\frac{6+9t}{\rho^2(1-t)} - \frac{1+t}{l_2^2-\rho^2} - \frac{3}{l_2^2-\rho^2 t}\right]\right\}. \qquad \text{(A16)}$$

Yet another z-differentiation of eqn (A11) results in

$$\frac{\partial^2}{\partial z^2}\iint_S \frac{B_2(N,N_0)}{R(M,N)}\,\mathrm{d}S_N = 2\pi\bar{t}^2\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{\frac{3\bar{t}^{1/2}}{(1-\bar{t})^{5/2}}\left[\tan^{-1}\left(\frac{\bar{t}^{1/2}}{(1-\bar{t})^{1/2}}\right)\right.\right.$$

$$\left.\left.-\tan^{-1}\left(\frac{\bar{t}^{1/2}(a^2-l_1^2)^{1/2}}{a(1-\bar{t})^{1/2}}\right)\right]-\frac{1}{(1-\bar{t})^2}\left[\frac{a(a^2-l_1^2)^{1/2}}{a^2-l_1^2\bar{t}}-2-\bar{t}+\frac{a(1+\bar{t})}{(a^2-l_1^2)^{1/2}}\right]+\frac{\rho^4 z(l_2^2+\rho^2\bar{t})(l_2^2-a^2)^{1/2}}{(l_2^2-\rho^2)(l_2^2-\rho^2\bar{t})^2(l_2^2-l_1^2)}\right\}. \tag{A17}$$

We apply the Λ-operator to the complex conjugate of eqn (A15), with the result

$$\Lambda\iint_S \bar{B}_2(N,N_0)\ln[R(M,N)+z]\,\mathrm{d}S_N = \pi t^2\rho\,\mathrm{e}^{\mathrm{i}\phi}\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{2(l_2^2-a^2)^{1/2}\left[\frac{l_1^2}{\rho^2 t}\right.\right.$$

$$\left.-\frac{2(\rho^2-l_1^2)}{\rho^2(1-t)^3}\right]-\frac{2[z^3-(l_2^2-a^2)^{3/2}]}{3\rho^2(1-t)^2}\left[t+4+\frac{6t}{1-t}\right]-2((l_2^2-a^2)^{1/2}-z)$$

$$\times\left[\frac{a^2-\rho^2 t}{\rho^2 t(1-t)^2}-\frac{a^2}{\rho^2(1-t)}\right]+\left[\frac{z}{\rho^2}-\frac{(a^2-l_1^2)(l_2^2-a^2)^{1/2}}{l_1^2(l_2^2-\rho^2 t)}\right]\left[\frac{a^2-\rho^2 t}{(1-t)^2}\right.$$

$$\left.-\frac{z^2 t}{(1-t)^3}\right]+\frac{z}{\rho^2[t(1-t)]^{1/2}}\left[\frac{3(a^2-\rho^2 t)}{(1-t)^2}-\frac{2a^2}{t(1-t)}\right.$$

$$\left.\left.-\frac{5z^2 t}{(1-t)^3}\right]\left[\tan^{-1}\left(\frac{t^{1/2}}{(1-t)^{1/2}}\right)-\tan^{-1}\left(\frac{t^{1/2}(a^2-l_1^2)^{1/2}}{a(1-t)^{1/2}}\right)\right]\right\}. \tag{A18}$$

Application of yet another Λ to eqn (A18) yields

$$\Lambda^2\iint_S \bar{B}_2(N,N_0)\ln[R(M,N)+z]\,\mathrm{d}S_N = \pi t^2\rho^2\,\mathrm{e}^{2\mathrm{i}\phi}\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{\frac{2(l_2^2-a^2)^{3/2}}{(l_2^2-l_1^2)\rho^2(1-t)^2}\left(4+t+\frac{6t}{1-t}\right)\right.$$

$$+\frac{8[z^3-(l_2^2-a^2)^{3/2}]}{3\rho^4(1-t)^2}\left(t-2+\frac{8+t}{(1-t)^2}\right)+\frac{2(l_2^2-a^2)^{1/2}}{\rho^2(l_2^2-l_1^2)}\left[\frac{l_1^2}{t}+\frac{a^2}{1-t}\right.$$

$$\left.-\frac{2(\rho^2-l_1^2)}{(1-t)^3}-\frac{a^2-\rho^2 t}{t(1-t)^2}\right]+4\left[\frac{(l_2^2-a^2)^{1/2}-z}{\rho^4}\right]\left[\frac{2(a^2-\rho^2 t)}{t(1-t)^3}-\frac{2a^2}{1-t}-\frac{a^2}{(1-t)^2}\right]$$

$$+\frac{4(l_2^2-a^2)^{1/2}}{\rho^2}\left[\frac{a^2-l_1^2}{l_2^2-l_1^2}\left(\frac{1}{t}+\frac{2}{(1-t)^3}\right)-\frac{2l_1^2}{\rho^2 t}-\frac{2}{(1-t)^3}+\frac{6(\rho^2-l_1^2)}{\rho^2(1-t)^4}\right]$$

$$+\frac{z}{\rho^4}\left[1-\frac{a(a^2-l_1^2)^{1/2}}{a^2-tl_1^2}\right]\left[\frac{6z^2 t}{(1-t)^4}-\frac{4(a^2-\rho^2 t)}{(1-t)^3}\right]+\left[\frac{a^2-\rho^2 t}{(1-t)^2}-\frac{z^2 t}{(1-t)^3}\right]$$

$$\times\left[\frac{za(a^2-l_1^2)^{1/2}}{\rho^2(a^2-tl_1^2)}\left(\frac{1}{l_2^2-l_1^2}+\frac{2a^2-2\rho^2 t(a^2-l_1^2)/(l_2^2-l_1^2)}{\rho^2(a^2-tl_1^2)}\right)-\frac{2z}{\rho^4}\right]-z\left[\frac{3(a^2-\rho^2 t)}{(1-t)^2}\right.$$

$$\left.-\frac{2a^2}{t(1-t)}-\frac{5z^2 t}{(1-t)^3}\right]\left[\frac{1}{\rho^4(1-t)}-\frac{a(a^2-l_1^2)^{1/2}}{\rho^2(a^2-tl_1^2)}\left(\frac{1}{\rho^2(1-t)}+\frac{1}{l_2^2-l_1^2}\right)\right]$$

$$+\frac{z}{\rho^4[t(1-t)]^{1/2}}\left[\frac{15(a^2-\rho^2 t)}{(1-t)^3}-\frac{6a^2}{t(1-t)^2}-\frac{35z^2 t}{(1-t)^4}\right]$$

$$\left.\times\left[\tan^{-1}\left(\frac{t^{1/2}(a^2-l_1^2)^{1/2}}{a(1-t)^{1/2}}\right)-\tan^{-1}\left(\frac{t^{1/2}}{(1-t)^{1/2}}\right)\right]\right\}. \tag{A19}$$

Formula (A19) looks too long, and I have not found a way to simplify it. On the other hand, the same result can be obtained by integration of eqn (A14) with respect to z. Such an integration can be performed by using the indefinite integrals from Appendix B, and at first glance it is too long and includes various trigonometric functions. Since eqn (A19) contains only $\tan^{-1}$ in the last line, then it may be concluded that the coefficients of all the other trigonometric functions should be zero. This simple idea led to a relatively short result, namely,

$$\Lambda^2\iint_S \bar{B}_2(N,N_0)\ln[R(M,N)+z]\,\mathrm{d}S_N = \pi t^2\rho^2\,\mathrm{e}^{2\mathrm{i}\phi}\frac{\sqrt{\rho_0^2-a^2}}{a^3}\left\{\frac{(l_2^2-a^2)^{1/2}}{\rho^2}\left[\frac{35t}{2(1-t)^4}\right.\right.$$

$$\left.+\frac{2(a^2-\rho^2 t)}{t(1-t)^2(l_2^2-\rho^2 t)}-\frac{l_1^2}{\rho^2}\left(\frac{8}{t}+\frac{35}{(1-t)^4}\right)\right]+\left[\frac{2[z^3-(l_2^2-a^2)^{3/2}]}{3\rho^2}\right.$$

$$
+(l_2^2-a^2)^{-1/2}\left[\frac{48-87t+38t^2+8t^3}{2\rho^2(1-t)^4}+[z-(l_2^2-a^2)^{1/2}]\left[\frac{a^2(-6+t-18t^2+8t^3)}{\rho^4 t(1-t)^3}\right.\right.
$$

$$
\left.-\frac{8-9t-2t^2}{\rho^2(1-t)^3}\right]+\frac{z}{\rho^4[t(1-t)]^{1/2}}\left[\frac{15(a^2-\rho^2 t)}{(1-t)^3}-\frac{6a^2}{t(1-t)^2}\right.
$$

$$
\left.\left.-\frac{35z^2 t}{(1-t)^4}\right]\left[\tan^{-1}\left(\frac{t^{1/2}(a^2-l_1^2)^{1/2}}{a(1-t)^{1/2}}\right)-\tan^{-1}\left(\frac{t^{1/2}z}{(1-t)^{1/2}}\right)\right]\right\}. \qquad \text{(A20)}
$$

As can be seen, eqn (A20) is much shorter than (A19), and except for the last line, looks totally different. Direct numerical computations show that (A19) and (A20) are identical, but since the author is in jail and has no access to contemporary computer facilities, he could not find a way to reduce (A19) to (A20) or to reduce both to a third expression which might be even simpler then (A20). This is left as an exercise for the reader.

It is of interest to note that eqn (A20) was obtained from (A14) by integration with respect to z. If we now differentiate (A20) with respect to z, we do not get (A14), we get something very different, namely,

$$
\Lambda^2\iint_S \frac{\bar{B}_2(N,N_0)}{R(M,N)}\,\mathrm{d}S_N=\pi t\rho^2 e^{i\varphi}\frac{\rho_0^2-a^2}{a^2}\left\{\frac{l_2(l_2^2-\rho^2)^{1/2}}{\rho^2(l_2^2-l_1^2)}\left[\frac{2(a^2-\rho^2 t)}{t(1-t)^2(l_2^2-\rho^2 t)}\right.\right.
$$

$$
\left.-\frac{l_1^2}{\rho^2}\left(\frac{8}{t}+\frac{35}{(1-t)^4}\right)-\frac{35t}{2(1-t)^4}\right]+\frac{z(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2}\left[-\frac{4(a^2-\rho^2 t)l_2^2}{\rho^2 t(1-t)^2(l_2^2-\rho^2 t)^2}\right.
$$

$$
\left.-\frac{2l_1^2}{\rho^4}\left(\frac{8}{t}-\frac{35}{(1-t)^2}\right)\right]+\left[\frac{2z}{\rho^2}\left(z-\frac{l_2(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2}\right)+\frac{l_2(l_2^2-\rho^2)^{1/2}}{l_2^2-l_1^2}\right]\frac{48-87t+38t^2+8t^3}{2\rho^2(1-t)^4}
$$

$$
+\left[1-\frac{l_2(l_2^2-\rho^2)^{1/2}}{l_2^2-l_1^2}\right]\left[\frac{8-9t-2t^2}{\rho^2(1-t)^3}+\frac{a^2(-6+t-18t^2+8t^3)}{\rho^4 t(1-t)^3}\right]
$$

$$
-\frac{z(l_2^2-a^2)^{1/2}}{\rho^2(l_2^2-l_1^2)(l_2^2-\rho^2 t)[t(1-t)]^{1/2}}\left[\frac{6a^2}{t(1-t)^2}-\frac{15(a^2-\rho^2 t)}{(1-t)^3}-\frac{35z^2 t}{(1-t)^4}\right]
$$

$$
+\frac{1}{\rho^4[t(1-t)]^{1/2}}\left[\frac{6a^2}{t(1-t)^2}-\frac{15(a^2-\rho^2 t)}{(1-t)^3}+\frac{105z^2 t}{(1-t)^4}\right]\left[\tan^{-1}\left(\frac{t^{1/2}z}{(1-t)^{1/2}}\right)-\tan^{-1}\left(\frac{t^{1/2}(a^2-l_1^2)^{1/2}}{a(1-t)^{1/2}}\right)\right]\right\}. \qquad \text{(A21)}
$$

Again, numerical computations show that eqn (A21) is identical to (A14), but no way was found to reduce one to the other or to reduce both to a third expression which would be simpler than both of them. Again the exercise is left to the reader.

Application of the Λ-operator to I_3 in eqn (A11) gives the result

$$
\Lambda\iint_S \frac{z B_2(N,N_0)}{R^3(M,N)}\,\mathrm{d}S_N=2\pi\frac{i\rho\, e^{i\varphi}\,\bar{t}^2 l_1^2(a^2+l_1^2\bar{t})}{a^2(a^2-l_1^2\bar{t})^2(l_2^2-l_1^2)}. \qquad \text{(A22)}
$$

Here $\bar{t}$ is defined in eqn (A29), and we used the property $\Lambda\bar{t}=0$. Yet another application of Λ to eqn (A20) yields

$$
\Lambda^3\iint_S \bar{B}_2(N,N_0)\ln[R(M,N)+z]\,\mathrm{d}S_N
$$

$$
=\pi t\rho^3 e^{3i\varphi}\frac{\rho_0^2-a^2}{a^2}\frac{(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2}\left\{\left[\frac{2(a^2-\rho^2 t)(2a^2-\rho^2 t-l_2^2)}{\rho^2 t(1-t)^2(l_2^2-\rho^2 t)^2}-\frac{(l_1^2-2a^2)}{\rho^4}\left(\frac{35}{(1-t)^4}+\frac{8}{t}\right)\right.\right.
$$

$$
\left.\left.-\frac{35t}{2\rho^2(1-t)^4}\right]-\frac{(l_2^2-a^2)}{\rho^2}\left[\frac{140t}{(1-t)^5}-\frac{8(a^2-\rho^2 t)}{t(1-t)^2(l_2^2-\rho^2 t)}-\frac{l_1^2}{\rho^2}\left(\frac{48}{t}+\frac{280}{(1-t)^5}\right)\right]\right]
$$

$$
+\left[\frac{(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2}\left(1-\frac{2(l_2^2-a^2)}{\rho^2}\right)-\frac{4[z^3-(l_2^2-a^2)^{3/2}]}{3\rho^2}\right]\frac{48+87t-38t^2+8t^3}{2\rho^2(1-t)^4}
$$

$$
+\left[\frac{2[z^3-(l_2^2-a^2)^{3/2}]}{3\rho^2}-z(l_2^2-a^2)^{1/2}\right]\frac{144-396t+190t^2-86t^3+16t^4}{\rho^4(1-t)^5}
$$

$$
+[z-(l_2^2-a^2)^{1/2}]\left[\frac{48-36t-4t^2}{\rho^4(1-t)^4}+\frac{a^2(-36+4t-36t^2)}{\rho^6 t(1-t)^4}\right]-\frac{(l_2^2-a^2)^{1/2}}{l_2^2-l_1^2}
$$

$$
+\frac{6[z-(l_2^2-a^2)^{1/2}]}{\rho^2}\frac{2-t}{1-t}\left[\frac{8-9t-2t^2}{\rho^2(1-t)^3}-\frac{a^2(-6+t-18t^2+8t^3)}{\rho^4 t(1-t)^3}\right]
$$

$$
-\frac{z}{\rho^2}\left[\frac{6a^2}{t(1-t)^2}-\frac{15(a^2-\rho^2 t)}{(1-t)^3}+\frac{35z^2 t}{(1-t)^4}\right]\left[\frac{a(a^2-l_1^2)^{1/2}}{\rho^2(a^2-tl_1^2)}\left(\frac{1}{\rho^2(1-t)}+\frac{1}{l_2^2-l_1^2}\right)\right.
$$

$$-\frac{1}{\rho^4(1-t)}\Bigg]-\frac{z}{\rho^6[t(1-t)]^{1/2}}\left[-\frac{30a^2}{t(1-t)^3}-\frac{105(a^2-\rho^2 t)}{(1-t)^4}\right.$$

$$\left.+\frac{315z^2 t}{(1-t)^5}\right]\left[\tan^{-1}\left(\frac{t^{1/2}}{(1-t)^{1/2}}\right)-\tan^{-1}\left(\frac{t^{1/2}(a^2-l_1^2)^{1/2}}{a(1-t)^{1/2}}\right)\right]\Bigg\}. \quad \text{(A23)}$$

It should be noted that

$$\Lambda \ln[R(M,N)+z] = \frac{\rho\,e^{i\phi}-r\,e^{i\psi}}{R(M,N)[R(M,N)+z]},$$

$$\Lambda^2 \ln[R(M,N)+z] = -\frac{(\rho\,e^{i\phi}-r\,e^{i\psi})^2[2R(M,N)+z]}{R^3(M,N)[R(M,N)+z]^2},$$

$$\Lambda^3 \ln[R(M,N)+z] = (\rho\,e^{i\phi}-r\,e^{i\psi})^3\frac{8R^2(M,N)+9zR(M,N)+3z^2}{R^5(M,N)[R(M,N)+z]^3}. \quad \text{(A24)}$$

Though I can not compute $L_1(M, N_0)$, as it is defined in eqn (32), I can compute all its derivatives. The simplest to compute is (Fabrikant, 1989)

$$J_1 = \iint_S \frac{zB_1(N,N_0)}{R^3(M,N)}\,dS_N = \int_0^{2\pi}\int_a^{\infty}\frac{1}{R(N,N_0)}\tan^{-1}\left(\frac{[(\rho_0^2-a^2)(r^2-a^2)]^{1/2}}{aR(N,N_0)}\right)\frac{zr\,dr\,d\psi}{R^3(M,N)}$$

$$=\frac{2\pi}{R(M,N_0)}\tan^{-1}\left(\frac{[(\rho_0^2-a^2)(l_2^2-a^2)]^{1/2}}{aR(M,N_0)}\right). \quad \text{(A25)}$$

The next integral to compute is

$$J_2 = \iint_S \frac{\rho\,e^{i\phi}-r\,e^{i\psi}}{R^3(M,N)}B_1(N,N_0)\,dS_N. \quad \text{(A26)}$$

The integral can be expressed through J_1 as follows

$$J_2 = \int_z^{\infty}\Lambda J_1\,dz, \quad \text{(A27)}$$

and it can be computed in the same way as it is done by Fabrikant (1989, Appendix A4.3), with the result

$$J_2 = \int_0^{2\pi}\int_a^{\infty}\frac{1}{R(N,N_0)}\tan^{-1}\left(\frac{[(\rho_0^2-a^2)(r^2-a^2)]^{1/2}}{aR(N,N_0)}\right)\frac{\rho\,e^{i\phi}-r\,e^{i\psi}}{R^3(M,N)}r\,dr\,d\psi$$

$$=\frac{2\pi}{\bar q}\left\{\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right)-\frac{z}{R_0}\tan^{-1}\left(\frac{j}{R_0}\right)+\frac{(\rho_0^2-a^2)^{1/2}}{\bar s}\left[\tan^{-1}\left(\frac{\bar s}{(a^2-l_1^2)^{1/2}}\right)-\tan^{-1}\left(\frac{\bar s}{a}\right)\right]\right\}. \quad \text{(A28)}$$

Here we introduced the notation

$$R_0 = R(M,N_0) = \sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2},\quad s=\sqrt{\rho\rho_0\,e^{i(\phi-\phi_0)}-a^2},$$

$$\bar s = \sqrt{\rho\rho_0\,e^{-i(\phi-\phi_0)}-a^2},\quad j=\sqrt{\rho_0^2-a^2}\sqrt{l_2^2-a^2}/a. \quad \text{(A29)}$$

Integration of eqn (A28) with respect to z yields

$$\int_0^{2\pi}\int_a^{\infty}\frac{1}{R(N,N_0)}\tan^{-1}\left(\frac{[(\rho_0^2-a^2)(r^2-a^2)]^{1/2}}{aR(N,N_0)}\right)\frac{(\rho\,e^{i\phi}-r\,e^{i\psi})r\,dr\,d\psi}{R(M,N)[R(M,N)+z]}$$

$$=\frac{2\pi}{\bar q}\left\{R_0\tan^{-1}\left(\frac{j}{R_0}\right)-z\tan^{-1}\left(\frac{\sqrt{\rho_0^2-a^2}}{a}\right)+\sqrt{\rho_0^2-a^2}\left[\sqrt{1-\bar\zeta}\tan^{-1}\left(\frac{a(1-\bar\zeta)^{1/2}}{(l_2^2-a^2)^{1/2}}\right)\right.\right.$$

$$\left.\left.-\frac{z}{\bar s}\left(\tan^{-1}\left(\frac{\bar s}{(a^2-l_1^2)^{1/2}}\right)-\tan^{-1}\left(\frac{\bar s}{a}\right)\right)\right]\right\}. \quad \text{(A30)}$$

Here the following indefinite integrals were used

$$\int \tan^{-1}\left(\frac{\bar{s}}{(a^2-l_1^2)^{1/2}}\right)\mathrm{d}z = z\tan^{-1}\left(\frac{\bar{s}}{(a^2-l_1^2)^{1/2}}\right)-\bar{s}\left[\sqrt{1-\zeta}\tan^{-1}\left(\frac{a(1-\zeta)^{1/2}}{(l_2^2-a^2)^{1/2}}\right)-\sin^{-1}\left(\frac{a}{l_2}\right)\right] \quad \text{(A31)}$$

$$\int \frac{z}{R_0}\tan^{-1}\left(\frac{j}{R_0}\right)\mathrm{d}z = R_0\tan^{-1}\left(\frac{j}{R_0}\right)-\sqrt{\rho_0^2-a^2}\left[\sqrt{1-\zeta}\tan^{-1}\left(\frac{a(1-\zeta)^{1/2}}{(l_2^2-a^2)^{1/2}}\right)\right.$$

$$\left.+\sqrt{1-\bar{\zeta}}\tan^{-1}\left(\frac{a(1-\bar{\zeta})^{1/2}}{(l_2^2-a^2)^{1/2}}\right)-\sin^{-1}\left(\frac{a}{l_2}\right)\right] \quad \text{(A32)}$$

and the identities

$$\rho_0^2(l_2^2-a^2\zeta)(l_2^2-a^2\bar{\zeta}) = a^2l_2^2(R_0^2+j^2),\quad [l_1^2-\rho\rho_0\,e^{i(\psi-\phi_0)}][l_1^2-\rho\rho_0\,e^{-i(\psi-\phi_0)}] = l_1^2(R_0^2+j^2). \quad \text{(A33)}$$

ζ was defined in expression (15). Application of the Λ-operator to eqn (A30) results in

$$\int_0^{2\pi}\int_a^{\infty}\frac{(\rho\,e^{i\phi}-r\,e^{i\psi})^2[2R(M,N)+z]}{R^3(M,N)[R(M,N)+z]^2}\tan^{-1}\left(\frac{[(\rho_0^2-a^2)(r^2-a^2)]^{1/2}}{aR(N,N_0)}\right)\frac{r\,\mathrm{d}r\,\mathrm{d}\psi}{R(N,N_0)}$$

$$= \frac{2\pi}{\bar{q}}\left\{\frac{R_0^2+z^2}{\bar{q}R_0}\tan^{-1}\left(\frac{j}{R_0}\right)-\frac{2z}{\bar{q}}\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right)\right.$$

$$-(\rho_0^2-a^2)^{1/2}\left[\frac{z}{\bar{s}}\left(\frac{2}{\bar{q}}+\frac{\rho_0\,e^{i\phi_0}}{\bar{s}^2}\right)\left(\tan^{-1}\left(\frac{\bar{s}}{(a^2-l_1^2)^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{s}}{a}\right)\right)\right.$$

$$\left.\left.-\frac{e^{i\phi}}{\rho_0(1-\bar{\zeta})^{1/2}}\tan^{-1}\left(\frac{a(1-\bar{\zeta})^{1/2}}{(l_2^2-a^2)^{1/2}}\right)\right]+\frac{za\,e^{i\phi}}{\rho\bar{s}^2}(a-(a^2-l_1^2)^{1/2})\right\}. \quad \text{(A34)}$$

The following identities were used here:

$$\Lambda\bar{s} = \frac{\rho_0\,e^{i\phi}}{\bar{s}},\quad \Lambda\sqrt{1-\bar{\zeta}} = -\frac{e^{i\phi}}{\rho_0(1-\bar{\zeta})^{1/2}},$$

$$\Lambda\tan^{-1}\left(\frac{a(1-\bar{\zeta})^{1/2}}{(l_2^2-a^2)^{1/2}}\right) = -\frac{(l_2^2-a^2)^{1/2}a\,e^{i\phi_0}}{\rho_0(1-\bar{\zeta})^{1/2}(l_2^2-a^2\bar{\zeta})}\left[1-\frac{\bar{q}\rho\,e^{i\phi}}{l_2^2-l_1^2}\right],$$

$$\Lambda\tan^{-1}\left(\frac{\bar{s}}{(a^2-l_1^2)^{1/2}}\right) = \frac{(a^2-l_1^2)^{1/2}}{\rho\rho_0\,e^{-i(\psi-\phi_0)}-l_1^2}\left[\frac{\rho_0\,e^{i\phi_0}}{\bar{s}}+\frac{\bar{s}\rho\,e^{i\phi}}{l_2^2-l_1^2}\right],$$

$$\Lambda\tan^{-1}\left(\frac{j}{R_0}\right) = \frac{R_0 j}{R_0^2+j^2}\left[\frac{\rho\,e^{i\phi}}{l_2^2-l_1^2}-\frac{q}{R_0^2}\right]. \quad \text{(A35)}$$

Applying yet another Λ-operator to eqn (A34), the result is

$$\int_0^{2\pi}\int_a^{\infty}\frac{(\rho\,e^{i\phi}-r\,e^{i\psi})^3[8R^2(M,N)+9zR(M,N)+3z^2]}{R^5(M,N)[R(M,N)+z]^3}\tan^{-1}\left(\frac{\sqrt{\rho_0^2-a^2}\sqrt{r^2-a^2}}{aR(N,N_0)}\right)\frac{r\,\mathrm{d}r\,\mathrm{d}\psi}{R(N,N_0)}$$

$$= \frac{2\pi}{\bar{q}}\left\{\frac{3R_0^4+6z^2R_0^2-z^4}{\bar{q}^2R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right)-\frac{8z}{\bar{q}^2}\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right)\right.$$

$$-(\rho_0^2-a^2)^{1/2}\left[\frac{3(1-\bar{\zeta})^{1/2}}{\bar{q}^2}\tan^{-1}\left(\frac{a(1-\bar{\zeta})^{1/2}}{(l_2^2-a^2)^{1/2}}\right)+\frac{z}{\bar{s}}\left(\frac{8}{\bar{q}^2}+\frac{4\rho_0\,e^{i\phi_0}}{\bar{q}\bar{s}^2}\right.\right.$$

$$\left.+\frac{3\rho_0^2\,e^{2i\phi_0}}{\bar{s}^4}\right)\left[\tan^{-1}\left(\frac{\bar{s}}{(a^2-l_1^2)^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{s}}{a}\right)\right]$$

$$\left.-\frac{z}{\bar{s}^2}\left(\frac{2}{\bar{q}}+\frac{\rho_0\,e^{i\phi_0}}{\bar{s}^2}\right)\left(\frac{(a^2-l_1^2)^{1/2}\rho_0\,e^{i\phi_0}}{\rho\rho_0\,e^{-i(\psi-\phi_0)}-l_1^2}-\frac{a\,e^{i\phi}}{\rho}\right)\right]+\frac{2za\,e^{i\phi}}{\rho\bar{s}^2}\left(\frac{1}{\bar{q}}+\frac{e^{i\phi}}{\rho}\right.$$

$$\left.\left.+\frac{\rho_0\,e^{i\phi_0}}{\bar{s}^2}\right)(a-(a^2-l_1^2)^{1/2})+\frac{j}{R_0^2+j^2}\left[\frac{\rho\,e^{3i\phi}\bar{q}}{l_2^2-l_1^2}-\frac{z^2q}{R_0\bar{q}}+\frac{e^{i\phi}(\rho^2-l_1^2)}{\bar{q}\rho}-2\,e^{2i\phi}\right]\right\}. \quad \text{(A36)}$$

Differentiation with respect to z of eqn (A25) leads to the integral

$$\frac{\partial}{\partial z}\iint_S \frac{zB_1(N,N_0)}{R^3(M,N)}\,\mathrm{d}S_N = \int_0^{2\pi}\int_a^{\infty}\left(\frac{1}{R^3(M,N)} - \frac{3z^2}{R^5(M,N)}\right)\tan^{-1}\left(\frac{\sqrt{\rho_0^2-a^2}\sqrt{r^2-a^2}}{aR(N,N_0)}\right)\frac{r\,\mathrm{d}r\,\mathrm{d}\psi}{R(N,N_0)}$$

$$= 2\pi\left\{-\frac{z}{R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right)+\frac{j}{z(R_0^2+j^2)}\left[\frac{l_2^2-\rho^2}{l_2^2-l_1^2}-\frac{z^2}{R_0^2}\right]\right\}. \quad \text{(A37)}$$

Application of the operator Λ to eqn (A25) yields

$$\int_0^{2\pi}\int_a^{\infty}\frac{3z(\rho\,\mathrm{e}^{\mathrm{i}\phi}-r\,\mathrm{e}^{\mathrm{i}\psi})}{R^5(M,N)}\tan^{-1}\left(\frac{\sqrt{\rho_0^2-a^2}\sqrt{r^2-a^2}}{aR(N,N_0)}\right)\frac{r\,\mathrm{d}r\,\mathrm{d}\psi}{R(N,N_0)}$$

$$= 2\pi\left\{\frac{q}{R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right)-\frac{j}{R_0^2+j^2}\left[\frac{\rho\,\mathrm{e}^{\mathrm{i}\phi}}{l_2^2-l_1^2}-\frac{q}{R_0^2}\right]\right\}. \quad \text{(A38)}$$

Yet another application of Λ-operator to eqn (A28) yields

$$\Lambda J_2 = -\int_0^{2\pi}\int_a^{\infty}\frac{3(\rho\,\mathrm{e}^{\mathrm{i}\phi}-r\,\mathrm{e}^{\mathrm{i}\psi})^2}{R^5(M,N)}\tan^{-1}\left(\frac{\sqrt{\rho_0^2-a^2}\sqrt{r^2-a^2}}{aR(N,N_0)}\right)\frac{r\,\mathrm{d}r\,\mathrm{d}\psi}{R(N,N_0)}$$

$$= \frac{2\pi}{\bar{q}}\left\{\frac{z(3R_0^2-z^2)}{\bar{q}R_0^3}\tan^{-1}\left(\frac{j}{R_0}\right)-\frac{2}{\bar{q}}\tan^{-1}\left(\frac{(\rho_0^2-a^2)^{1/2}}{a}\right)+\frac{zj}{R_0^2+j^2}\left[\frac{q}{R_0^2}\right.\right.$$

$$\left.-\frac{\bar{q}\rho^2\,\mathrm{e}^{2\mathrm{i}\phi}}{(l_2^2-l_1^2)(\rho^2-l_1^2)}\right]-\frac{(\rho_0^2-a^2)^{1/2}}{\bar{s}}\left(\frac{2}{\bar{q}}+\frac{\rho_0\,\mathrm{e}^{\mathrm{i}\phi_0}}{\bar{s}^2}\right)\left[\tan^{-1}\left(\frac{\bar{s}}{(a^2-l_1^2)^{1/2}}\right)\right.$$

$$\left.\left.-\tan^{-1}\left(\frac{\bar{s}}{a}\right)\right]+\frac{(\rho_0^2-a^2)^{1/2}}{\bar{s}^2}\left[\frac{(a^2-l_1^2)^{1/2}\rho_0\,\mathrm{e}^{\mathrm{i}\phi_0}}{\rho\rho_0\,\mathrm{e}^{-\mathrm{i}(\phi-\phi_0)}-l_1^2}-\frac{a\,\mathrm{e}^{\mathrm{i}\phi}}{\rho}\right]\right\}. \quad \text{(A39)}$$

The next integral to compute is ΛI_1, and from (A9)

$$\Lambda\iint_S \frac{B_2(N,N_0)}{R(M,N)}\,\mathrm{d}S_N = 2\pi\frac{\bar{l}^2}{a^3}\rho\,\mathrm{e}^{\mathrm{i}\phi}(\rho_0^2-a^2)^{1/2}\left\{\frac{1}{1-\bar{l}}\left[1-\frac{a(a^2-l_1^2)^{1/2}}{a^2-l_1^2\bar{l}}\right]\right.$$

$$\left.+\frac{\bar{l}^{1/2}}{(1-\bar{l})^{3/2}}\left[\tan^{-1}\left(\frac{\bar{l}^{1/2}}{(1-\bar{l})^{1/2}}\right)-\tan^{-1}\left(\frac{\bar{l}^{1/2}(a^2-l_1^2)^{1/2}}{a(1-\bar{l})^{1/2}}\right)\right]\right\}. \quad \text{(A40)}$$

Applying the Λ-operator to (A40), results in

$$\Lambda^2\iint_S \frac{B_2(N,N_0)}{R(M,N)}\,\mathrm{d}S_N = 2\pi\frac{\rho^2\,\mathrm{e}^{2\mathrm{i}\phi}(\rho_0^2-a^2)^{1/2}(a^2-l_1^2)^{1/2}\bar{l}^2(a^2+l_1^2\bar{l})}{a^2(l_2^2-l_1^2)(a^2-l_1^2\bar{l})^2}. \quad \text{(A41)}$$

Integration with respect to z of both sides of eqn (A41) yields

$$\Lambda^2\iint_S B_2(N,N_0)\ln[R(M,N)+z]\,\mathrm{d}S_N = 2\pi\frac{\bar{l}^2}{a^2}\rho^2\,\mathrm{e}^{2\mathrm{i}\phi}(\rho_0^2-a^2)^{1/2}\left[\frac{\bar{l}l_1(\rho^2-l_1^2)^{1/2}}{(a^2-l_1^2\bar{l})(a^2-\rho^2\bar{l})}\right.$$

$$\left.-\frac{a}{(a^2-\rho^2\bar{l})^{3/2}}\tan^{-1}\left(\frac{(a^2-\rho^2\bar{l})^{1/2}}{(l_2^2-a^2)^{1/2}}\right)\right]. \quad \text{(A42)}$$

Application of yet another Λ to (A42) gives

$$\Lambda^3\iint_S B_2(N,N_0)\ln[R(M,N)+z]\,\mathrm{d}S_N$$

$$= 2\pi\frac{\bar{l}^3}{a}\rho^3\,\mathrm{e}^{3\mathrm{i}\phi}(\rho_0^2-a^2)^{1/2}\left\{\frac{(l_2^2-a^2)^{1/2}}{(a^2-l_1^2\bar{l})(a^2-\rho^2\bar{l})}\left[\frac{a^2+2\rho^2\bar{l}}{l_2^2(a^2-\rho^2\bar{l})}\right.\right.$$

$$\left.\left.+\frac{a^2+\rho^2\bar{l}}{l_2^2\bar{l}(l_2^2-l_1^2)}+\frac{2(a^2-l_1^2)}{(l_2^2-l_1^2)(a^2-l_1^2\bar{l})}\right]-\frac{3}{(a^2-\rho^2\bar{l})^{5/2}}\tan^{-1}\left(\frac{(a^2-\rho^2\bar{l})^{1/2}}{(l_2^2-a^2)^{1/2}}\right)\right\}. \quad \text{(A43)}$$

The following identity was used here

$$\Lambda\tan^{-1}\left(\frac{(a^2-\rho^2\bar{l})^{1/2}}{(l_2^2-a^2)^{1/2}}\right) = -\frac{\rho\,\mathrm{e}^{\mathrm{i}\phi}a^2(l_2^2-a^2)^{1/2}}{l_2^2(a^2-\rho^2\bar{l})^{1/2}(a^2-l_1^2\bar{l})}\left[\bar{l}+\frac{a^2-\rho^2\bar{l}}{l_2^2-l_1^2}\right]. \quad \text{(A44)}$$

APPENDIX B

For the readers' convenience, I present here some regular integrals which are not explicitly present in the tables

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}(b^2-x^2)} = \frac{1}{b\sqrt{b^2-a^2}}\tan^{-1}\left(\frac{b\sqrt{a^2-x^2}}{x\sqrt{b^2-a^2}}\right), \tag{B1}$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}(b^2-x^2)^2} = -\frac{2b^2-a^2}{2b^3(b^2-a^2)^{3/2}}\tan^{-1}\left(\frac{b\sqrt{a^2-x^2}}{x\sqrt{b^2-a^2}}\right) - \frac{x\sqrt{a^2-x^2}}{2b^2(b^2-x^2)(b^2-a^2)}, \tag{B2}$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-\rho^2}(x^2-\alpha^2)} = \frac{1}{\alpha\sqrt{\rho^2-\alpha^2}}\tan^{-1}\left(\frac{\alpha\sqrt{x^2-\rho^2}}{x\sqrt{\rho^2-\alpha^2}}\right), \tag{B3}$$

$$\int \frac{\mathrm{d}x}{(x^2-\rho^2)^{3/2}(x^2-\alpha^2)} = -\frac{1}{\alpha(\rho^2-\alpha^2)^{3/2}}\tan^{-1}\left(\frac{\alpha\sqrt{x^2-\rho^2}}{x\sqrt{\rho^2-\alpha^2}}\right) - \frac{x}{\rho^2(\rho^2-\alpha^2)\sqrt{x^2-\rho^2}}, \tag{B4}$$

$$\int \frac{\mathrm{d}x}{(x^2-\rho^2)^{3/2}(x^2-\alpha^2)^2} = \frac{\rho^2-4\alpha^2}{2\alpha^3(\rho^2-\alpha^2)^{5/2}}\tan^{-1}\left(\frac{\alpha\sqrt{x^2-\rho^2}}{x\sqrt{\rho^2-\alpha^2}}\right) - \frac{x}{(\rho^2-\alpha^2)^2}\left[\frac{\sqrt{x^2-\rho^2}}{2\alpha^2(x^2-\alpha^2)} + \frac{1}{\rho^2\sqrt{x^2-\rho^2}}\right], \tag{B5}$$

$$\int \frac{\mathrm{d}x}{(x^2-\rho^2)^{5/2}(x^2-\alpha^2)} = \frac{1}{\alpha(\rho^2-\alpha^2)^{5/2}}\tan^{-1}\left(\frac{\alpha\sqrt{x^2-\rho^2}}{x\sqrt{\rho^2-\alpha^2}}\right) + \frac{x}{3\rho^2(\rho^2-\alpha^2)\sqrt{x^2-\rho^2}}\left(\frac{2}{\rho^2} + \frac{3}{\rho^2-\alpha^2} - \frac{1}{x^2-\rho^2}\right), \tag{B6}$$

$$\int \frac{\mathrm{d}x}{(x^2-\rho^2)^{5/2}(x^2-\alpha^2)^2} = \frac{6\alpha^2-\rho^2}{2\alpha^3(\rho^2-\alpha^2)^{7/2}}\tan^{-1}\left(\frac{\alpha\sqrt{x^2-\rho^2}}{x\sqrt{\rho^2-\alpha^2}}\right)$$
$$+\frac{x}{(\rho^2-\alpha^2)\sqrt{x^2-\rho^2}}\left[\frac{\rho^2+4\alpha^2}{2\alpha^2\rho^2(\rho^2-\alpha^2)} - \frac{1}{2\alpha^2(x^2-\alpha^2)} + \frac{2x^2-3\rho^2}{3\rho^4(x^2-\rho^2)}\right], \tag{B7}$$

$$\int \frac{\mathrm{d}x}{(x^2-\rho^2)^{7/2}(x^2-\alpha^2)} = -\frac{1}{\alpha(\rho^2-\alpha^2)^{7/2}}\tan^{-1}\left(\frac{\alpha\sqrt{x^2-\rho^2}}{x\sqrt{\rho^2-\alpha^2}}\right)$$
$$+\frac{x}{15\rho^2(\rho^2-\alpha^2)\sqrt{x^2-\rho^2}}\left[\frac{5}{(\rho^2-\alpha^2)(x^2-\rho^2)} - \frac{3}{(x^2-\rho^2)^2} + \frac{4}{\rho^2(x^2-\rho^2)}\right.$$
$$\left. - \frac{10}{\rho^2(\rho^2-\alpha^2)} - \frac{8}{\rho^4} - \frac{15}{(\rho^2-\alpha^2)^2}\right], \tag{B8}$$

$$\int \frac{\mathrm{d}x}{(x^2-\rho^2)^{7/2}(x^2-\alpha^2)^2} = \frac{\rho^2-8\alpha^2}{2\alpha^3(\rho^2-\alpha^2)^{9/2}}\tan^{-1}\left(\frac{\alpha\sqrt{x^2-\rho^2}}{x\sqrt{\rho^2-\alpha^2}}\right)$$
$$-\frac{x\sqrt{x^2-\rho^2}}{2\alpha^2(x^2-\alpha^2)(\rho^2-\alpha^2)^4} + \frac{x}{15\rho^2(\rho^2-\alpha^2)^3\sqrt{x^2-\rho^2}}\left[\frac{10}{(\rho^2-\alpha^2)(x^2-\rho^2)}\right.$$
$$\left. - \frac{3}{(x^2-\rho^2)^2} + \frac{4}{\rho^2(x^2-\rho^2)} - \frac{20}{\rho^2(\rho^2-\alpha^2)} - \frac{8}{\rho^4} - \frac{45}{(\rho^2-\alpha^2)^2}\right]. \tag{B9}$$

I present below some indefinite integrals involving l_1 and l_2 which were used in this paper

$$\int \tan^{-1}(c\sqrt{a^2-l_1^2})\,\mathrm{d}z = z\tan^{-1}(c\sqrt{a^2-l_1^2}) + \frac{1}{c}\left[\sin^{-1}\left(\frac{a}{l_2}\right) - \frac{\sqrt{1+c^2(a^2-\rho^2)}}{\sqrt{1+c^2a^2}}\sin^{-1}\left(\frac{a\sqrt{1+c^2(a^2-\rho^2)}}{l_2\sqrt{1+c^2(a^2-l_1^2)}}\right)\right]. \tag{B10}$$

Here c does not depend on z

$$\int \tan^{-1}\left(\frac{\sqrt{t}\sqrt{a^2-l_1^2}}{a\sqrt{1-t}}\right)\mathrm{d}z = z\tan^{-1}\left(\frac{\sqrt{t}\sqrt{a^2-l_1^2}}{a\sqrt{1-t}}\right) + \frac{a\sqrt{1-t}}{\sqrt{t}}\left[\sin^{-1}\left(\frac{a}{l_2}\right) - \frac{\sqrt{a^2-\rho^2 t}}{a}\sin^{-1}\left(\frac{\sqrt{a^2-\rho^2 t}}{\sqrt{l_2^2-\rho^2 t}}\right)\right], \tag{B11}$$

$$\int z^2 \tan^{-1}(c\sqrt{a^2-l_1^2})\,\mathrm{d}z = \frac{z^3}{3}\tan^{-1}(c\sqrt{a^2-l_1^2})$$

$$-\frac{1}{3c}\left\{\sqrt{\rho^2-l_1^2}\left[\frac{l_1}{2}+\frac{a^2\rho^2c^2}{l_1(1+a^2c^2)}\right]+\left[\frac{\rho^2}{2}-\frac{1+c^2(a^2-\rho^2)}{c^2(1+a^2c^2)}\right.\right.$$

$$\left.\left.+\frac{a^2\rho^2c^2}{1+a^2c^2}\right]\sin^{-1}\left(\frac{a}{l_2}\right)+\frac{[1+c^2(a^2-\rho^2)]^{3/2}}{c^2(1+c^2a^2)^{3/2}}\sin^{-1}\left(\frac{a\sqrt{1+c^2(a^2-\rho^2)}}{l_2\sqrt{1+c^2(a^2-l_1^2)}}\right)\right\}, \quad \text{(B12)}$$

$$\int z^2\tan^{-1}\left(\frac{\sqrt{t}\sqrt{a^2-l_1^2}}{a\sqrt{1-t}}\right)\mathrm{d}z = \frac{z^3}{3}\tan^{-1}\left(\frac{\sqrt{t}\sqrt{a^2-l_1^2}}{a\sqrt{1-t}}\right)-\frac{a\sqrt{1-t}}{3\sqrt{t}}\left\{\sqrt{\rho^2-l_1^2}\left(\frac{l_1}{2}+\frac{\rho^2 t}{l_1}\right)\right.$$

$$\left.+\left(a^2+\tfrac{3}{2}\rho^2-\frac{a^2}{t}\right)\sin^{-1}\left(\frac{a}{l_2}\right)+\frac{(a^2-\rho^2 t)^{3/2}(1-t)}{at}\sin^{-1}\left(\frac{\sqrt{a^2-\rho^2 t}}{\sqrt{l_2^2-\rho^2 t}}\right)\right\}, \quad \text{(B13)}$$

$$\int\frac{\mathrm{d}z}{\sqrt{a^2-l_1^2}} = -\sin^{-1}\left(\frac{a}{l_2}\right)+\frac{\sqrt{l_2^2-a^2}}{a}-\frac{\sqrt{a^2-\rho^2}}{a}\tan^{-1}\left(\frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-\rho^2}}\right), \quad \text{(B14)}$$

$$\int\sqrt{a^2-l_1^2}\,\mathrm{d}z = \frac{2a^2-l_1^2}{2a}\sqrt{l_2^2-a^2}+\frac{\rho^2}{2}\sin^{-1}\left(\frac{a}{l_2}\right), \quad \text{(B15)}$$

$$\int\frac{z^2\,\mathrm{d}z}{\sqrt{a^2-l_1^2}} = \frac{(l_2^2-a^2)^{3/2}}{3a}+\frac{l_1^2\sqrt{l_2^2-a^2}}{2a}+\frac{\rho^2}{2}\sin^{-1}\left(\frac{a}{l_2}\right), \quad \text{(B16)}$$

$$\int\frac{\sqrt{a^2-l_1^2}}{1-c^2l_1^2}\,\mathrm{d}z = -\frac{1}{c^2}\sin^{-1}\left(\frac{a}{l_2}\right)+a\sqrt{l_2^2-a^2}-\frac{1-c^4a^2\rho^2}{c^2\sqrt{1-c^2\rho^2}}\tan^{-1}\left(\frac{\sqrt{l_2^2-a^2}}{a\sqrt{1-c^2\rho^2}}\right), \quad \text{(B17)}$$

$$\int\frac{z^2\sqrt{a^2-l_1^2}}{b^2-l_1^2}\,\mathrm{d}z = \sqrt{\rho^2-l_1^2}\left[\frac{l_1}{2}-\frac{a^2\rho^2(b^2-a^2)}{b^4 l_1}\right]+\left(\frac{\rho^2}{2}+a^2-b^2\right)\sin^{-1}\left(\frac{a}{l_2}\right)$$

$$+\frac{a(l_2^2-a^2)^{3/2}}{3b^2}+\frac{(b^2-a^2)(b^4-a^2\rho^2)}{b^5}\sqrt{b^2-\rho^2}\sin^{-1}\left(\frac{l_1\sqrt{b^2-\rho^2}}{\rho\sqrt{b^2-l_1^2}}\right). \quad \text{(B18)}$$

Here b and c are the quantities which do not depend on z.

$$\int\frac{z^2\sqrt{a^2-l_1^2}}{a^2-l_1^2 t}\,\mathrm{d}z = \sqrt{\rho^2-l_1^2}\left(\frac{l_1}{2t}-\frac{\rho^2(1-t)}{l_1}\right)+\frac{1}{t}\left(\frac{\rho^2}{2}+a^2-\frac{a^2}{t}\right)\sin^{-1}\left(\frac{a}{l_2}\right)$$

$$+\frac{(l_2^2-a^2)^{3/2}}{3a}+\sqrt{a^2-\rho^2 t}\,\frac{(1-t)(a^2-\rho^2t^2)}{at^2}\sin^{-1}\left(\frac{\sqrt{a^2-\rho^2 t}}{\sqrt{l_2^2-\rho^2 t}}\right), \quad \text{(B19)}$$

$$\int\frac{z^2(l_2^2+c^2)(l_2^2-a^2)\,\mathrm{d}z}{l_2(l_2^2-c^2)^2(l_2^2-\rho^2)^{5/2}(l_2^2-l_1^2)} = \left(\frac{c^2-3a^2}{c^4\rho^4}-\frac{2a^2}{c^2\rho^6}\right)\left[\sqrt{l_2^2-a^2}-a\cos^{-1}\left(\frac{a}{l_2}\right)\right]$$

$$-\frac{a^2}{c^2\rho^4}\left[\frac{1}{2a}\cos^{-1}\left(\frac{a}{l_2}\right)-\frac{\sqrt{l_2^2-a^2}}{2l_2^2}\right]+\left[\frac{3a^2-c^2}{c^2}-\frac{4(a^2-c^2)}{\rho^2-c^2}\right]\Bigg[\sqrt{l_2^2-a^2}$$

$$-\sqrt{a^2-c^2}\tan^{-1}\left(\frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-c^2}}\right)\Bigg]\frac{1}{c^2(\rho^2-c^2)^2}+\frac{2(a^2-c^2)}{c^2(\rho^2-c^2)^2}\Bigg[\frac{\sqrt{l_2^2-a^2}}{2(l_2^2-c^2)}$$

$$-\frac{1}{2\sqrt{a^2-c^2}}\tan^{-1}\left(\frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-c^2}}\right)\Bigg]+\Bigg[2\rho^2-a^2+c^2-2(\rho^2+c^2)(\rho^2-a^2)\Bigg(\frac{1}{\rho^2}$$

$$+\frac{1}{\rho^2-c^2}\Bigg)\Bigg]\frac{1}{\rho^4(\rho^2-c^2)^2}\left[\sqrt{l_2^2-a^2}-\sqrt{a^2-\rho^2}\tan^{-1}\left(\frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-\rho^2}}\right)\right]$$

$$+\frac{(\rho^2+c^2)(\rho^2-a^2)}{\rho^4(\rho^2-c^2)^2}\left[\frac{1}{2\sqrt{a^2-\rho^2}}\tan^{-1}\left(\frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-\rho^2}}\right)-\frac{\sqrt{l_2^2-a^2}}{2(l_2^2-\rho^2)}\right], \quad \text{(B20)}$$

$$\int\frac{a^3\,\mathrm{d}z}{l_2^2(a^2-l_1^2)^{3/2}} = \int\frac{l_2\,\mathrm{d}z}{(l_2^2-\rho^2)^{3/2}} = -\frac{a}{\rho^2}\cos^{-1}\left(\frac{a}{l_2}\right)+\frac{2a^2+\rho^2}{2\rho^2\sqrt{a^2-\rho^2}}\tan^{-1}\left(\frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-\rho^2}}\right)-\frac{\sqrt{l_2^2-a^2}}{2(l_2^2-\rho^2)}, \quad \text{(B21)}$$

$$\int \frac{a^3 l_1^2 \,\mathrm{d}z}{l_2^2(a^2-l_1^2)^{3/2}} = \int \frac{l_2 l_1^2 \,\mathrm{d}z}{(l_2^2-\rho^2)^{3/2}} = -\frac{a(4a^2+\rho^2)}{2\rho^2}\cos^{-1}\left(\frac{a}{l_2}\right)$$

$$-\frac{a^2}{2}\sqrt{l_2^2-a^2}\left(\frac{1}{l_2^2}+\frac{1}{l_2^2-\rho^2}\right)+\frac{a^2}{\sqrt{a^2-\rho^2}}\,\frac{4a^2-\rho^2}{2\rho^2}\tan^{-1}\left(\frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-\rho^2}}\right), \quad \text{(B22)}$$

$$\int \frac{\sqrt{a^2-l_1^2}\,l_2^2 z^2\,\mathrm{d}z}{l_2^2-\alpha^2} = a\left\{\sqrt{l_2^2-a^2}\left(\alpha^2-\rho^2+\frac{l_1^2\rho^2}{2\alpha^2}+\frac{l_2^2-a^2}{3}\right)-\frac{a\rho^2}{\alpha^2}\left[\frac{a^2(\alpha^2-\rho^2)}{\alpha^2}\right.\right.$$

$$\left.\left.+\frac{\rho^2}{2}\right]\cos^{-1}\left(\frac{a}{l_2}\right)-\frac{(\alpha^2-\rho^2)(\alpha^4-a^2\rho^2)}{\alpha^4}\sqrt{a^2-\alpha^2}\tan^{-1}\left(\frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-\alpha^2}}\right)\right\}, \quad \text{(B23)}$$

$$\int \frac{\mathrm{d}z}{\sqrt{a^2-l_1^2}(l_2^2-\alpha^2)} = -\frac{1}{a}\left\{\frac{a}{\alpha^2}\cos^{-1}\left(\frac{a}{l_2}\right)+\frac{\sqrt{a^2-\rho^2}}{\rho^2-\alpha^2}\tan^{-1}\left(\frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-\rho^2}}\right)\right.$$

$$\left.+\frac{\alpha^4-a^2\rho^2}{\alpha^2(\rho^2-\alpha^2)\sqrt{a^2-\alpha^2}}\tan^{-1}\left(\frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-\alpha^2}}\right)\right\}, \quad \text{(B24)}$$

$$\int \frac{z^2\,\mathrm{d}z}{\sqrt{a^2-l_1^2}(l_2^2-\rho^2)} = \frac{1}{a}\left\{\sqrt{l_2^2-a^2}\left(1-\frac{a^2}{2l_2^2}\right)+a\left(\frac{1}{2}-\frac{a^2}{\rho^2}\right)\cos^{-1}\left(\frac{a}{l_2}\right)+\frac{(a^2-\rho^2)^{3/2}}{\rho^2}\tan^{-1}\left(\frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-\rho^2}}\right)\right\},$$

$$\text{(B25)}$$

$$\int \frac{\sqrt{a^2-l_1^2}(a^2+l_1^2 t)}{(l_2^2-l_1^2)(a^2-l_1^2 t)^2}\,\mathrm{d}z = \frac{a}{(a^2-\rho^2 t)^{3/2}}\tan^{-1}\left(\frac{\sqrt{l_2^2-a^2}}{\sqrt{a^2-\rho^2 t}}\right)+\frac{t l_1\sqrt{\rho^2-l_1^2}}{(a^2-l_1^2 t)(a^2-\rho^2 t)}. \quad \text{(B26)}$$

All the integrals involving l_1 and l_2 were computed by using the substitutions

$$z = \frac{\sqrt{a^2-l_1^2}\sqrt{\rho^2-l_1^2}}{l_1} \quad \text{or} \quad z = \frac{\sqrt{l_2^2-a^2}\sqrt{l_2^2-\rho^2}}{l_2},$$

$$\mathrm{d}z = -\frac{l_2^2-l_1^2}{z l_1}\,\mathrm{d}l_1 \quad \text{or} \quad \mathrm{d}z = \frac{l_2^2-l_1^2}{z l_2}\,\mathrm{d}l_2. \quad \text{(B27)}$$